Kings on the Diagonal

DA – HW1 – Problem #29

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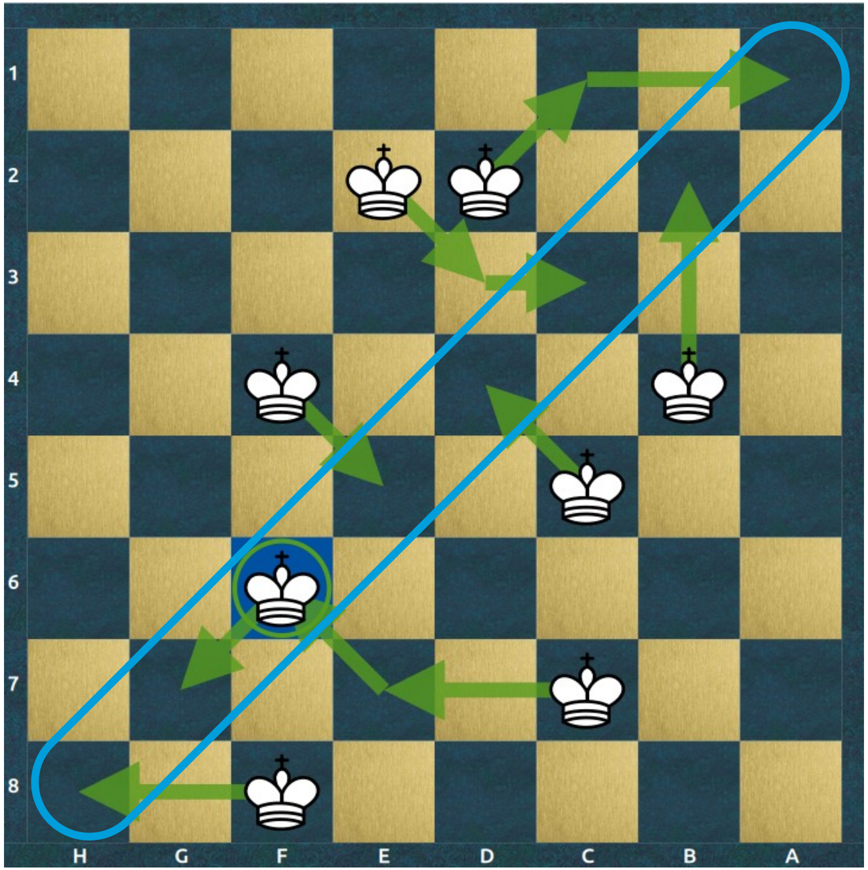
Problem Statement

- We have an *nxn* chess board and *n* kings at locations (x_i, y_i).
- Find the minimum number of moves by which all the kings are placed on the main diagonal.
- Target: *O(nlogn)*



What we actually want

- Find a destination (on the diagonal) for each king.
  - For king *k*, determine *i* to move the king on (x_k, y_k) to (i,i) on diagonal.
- Such that the total moves are minimized.

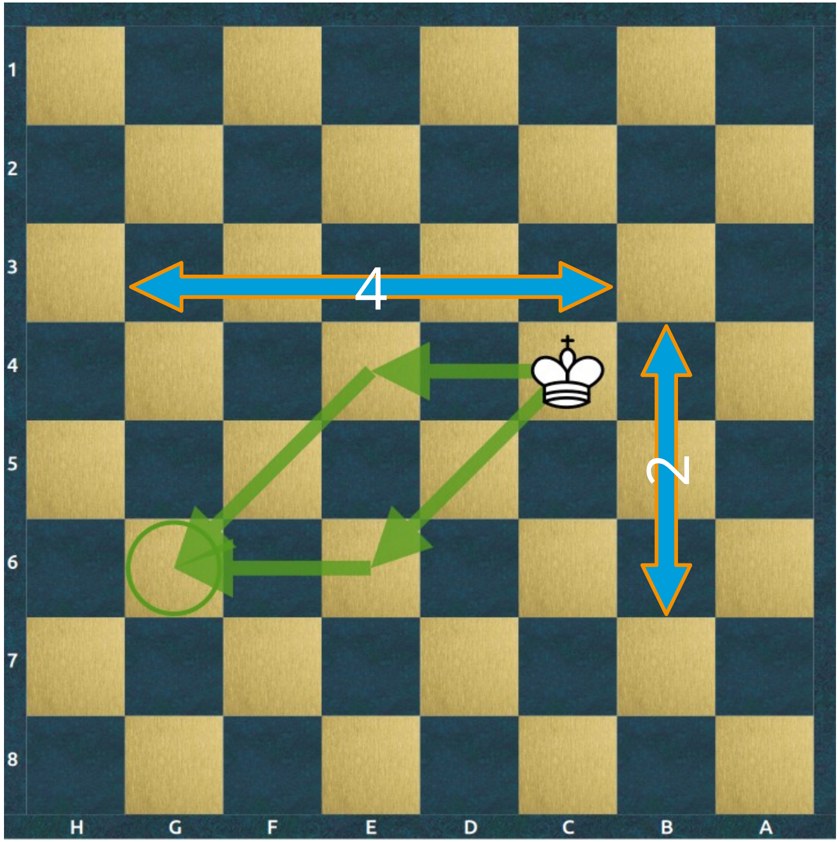
$$minimize \sum_{k=0}^{n} dist\{(x_k, y_k), (i, i)\}$$

- Now we need to define the distance.

What is the distance here

- A king moves one step to any of the 8 neighbor squares
- So it move from (x_1,y_1) to (x_2,y_2) by $\mathbf{max(|x_1-x_2|,|y_1-y_2|)}$ moves

(called Chebyshev distance)



What we actually want (cont.)

- So the distance between a king at *(x_k,y_k)* and a destination *(i,i)* on the main diagonal is calculated as follows:

$$dist\{(x_k, y_k),(i,i)\} = max\{|x_k - i|, |y_k - i|\} = \frac{|(x_k + y_k) - 2i| + |x_k - y_k|}{2}$$

- If we define $u_k = x_k + y_k$ and $v_k = x_k - y_k$, our objective will be to minimize the total distance:

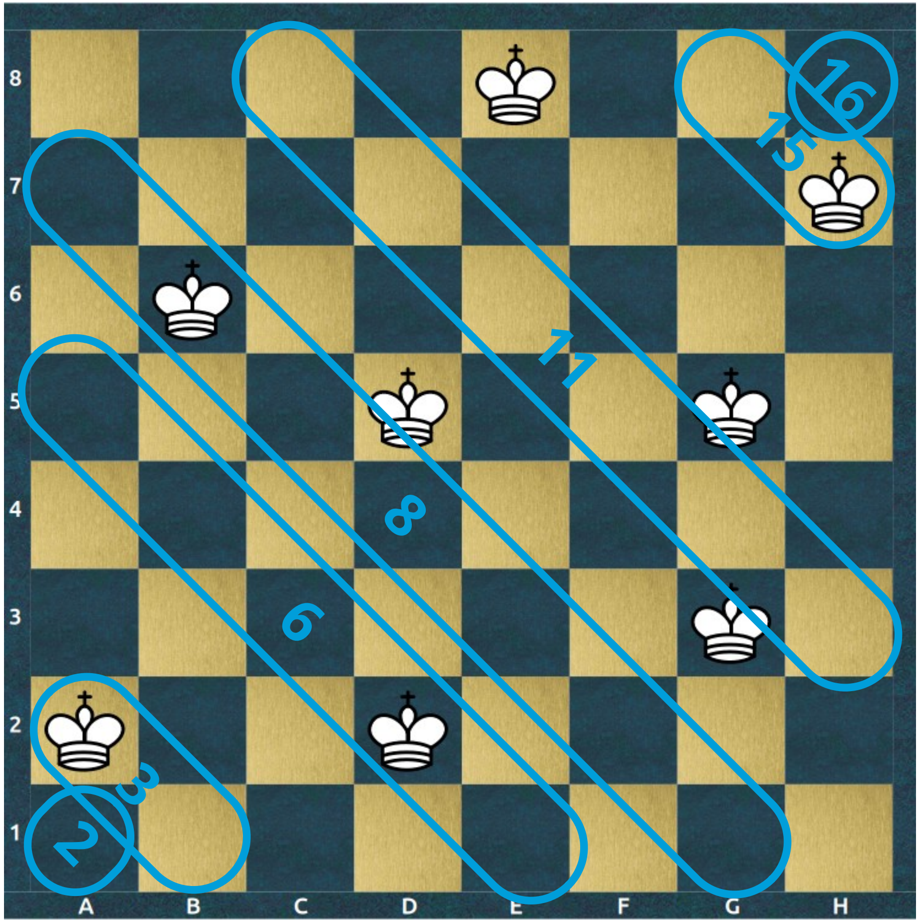
$$minimize \frac{1}{2}\sum_{k=1}^{n}|v_k| + \frac{1}{2}\sum_{k=1}^{n}|u_k - 2i| \Rightarrow \boxed{minimize \sum_{k=1}^{n}|u_k - 2i|}$$

Let's take a closer look

- So, our objective is to find an i for each k to minimize

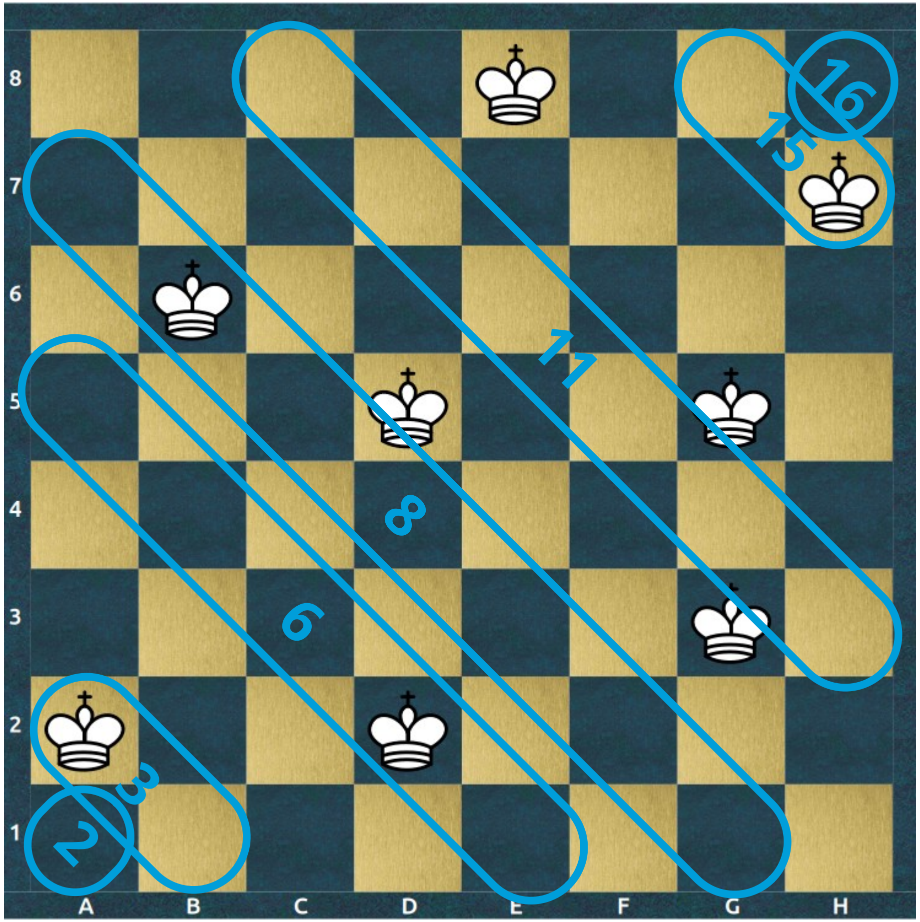
$$\sum_{k=1}^{n} |u_k - 2i|$$

- So let's see what's the u_k (sum of coordinates) for some squares on the board:



Some thinking…

- Ideally, $\sum_{k=1}^{n} |u_k - 2i|$ would be zero if we could send each king to a square on the diagonal in the same circle ($u_k=2i$).
- But that's not possible:
  - White-colored circles (odd u_k) don't include diagonal squares.
  - Multiple kings may be placed in the same circle.



And finally the solution!

- The previous slide gave us an intuition to match smaller u_k's with smaller i's ⇒ **Sorting kings based on u_k!**
- So the final algorithm would be:
  - Calculate $u_k = x_k + y_k$ for each king.
  - Sort the kings based on their u_k: show the i^{th} king by $u_{(i)}$
  - Match $u_{(i)}$ with the diagonal square (i,i), such that the minimum number of moves will be:

$$\frac{1}{2}\sum_{i=1}^{n} |v_{(i)}| + |u_{(i)} - 2i|$$

Execution time

- In order to calculate $\frac{1}{2}\sum_{i=1}^{n}\left|v_{(i)}\right|+\left|u_{(i)}-2i\right|$, we only need to sort the kings based on their u_k.
- And sorting can be done in ***O(nlogn)***.

Notes

- Proving that assigning u_k's to i's in an increasing order minimizes their sum of absolute differences, can be easily done using the lemma $|a-c|+|b-d| \leq |a-d|+|b-c|$.
- How we assign kings with similar u_k's (i.e., in the same circle) to i's doesn't affect the total #moves.