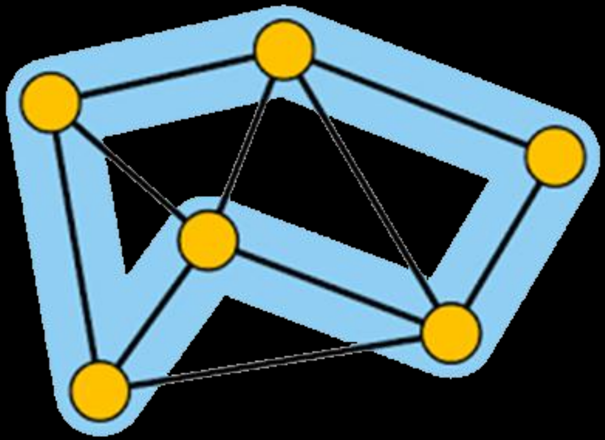
Hamilton Cycle Problem

specifying if graph has a Hamilton Cycle or not



Type: Dynamic Programming

صورت سوال

سوال ۳۴ — دور همیلتونی

صورت سوال

یک گراف ساده n راسی داریم. الگوریتمی با پیچیدگی زمانی $O(2^n n^2)$ ارائه دهید که مشخص کند آیا گراف دوری دارد که شامل تمام رئوس گراف باشد یا خیر.

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Hamilton loop Defenition

Consider Graph G(V,E):

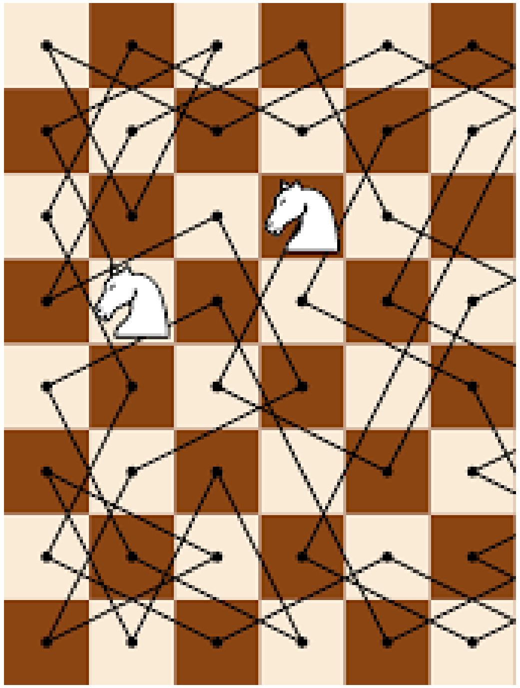
Graph G would contain a Hamilton Cycle IF:

- Graph G Should Contain At least **A path with length |V|+1**
- The path should cover every vertex **only 1 time**
- Finally the path should end with the **Starting Vertex**

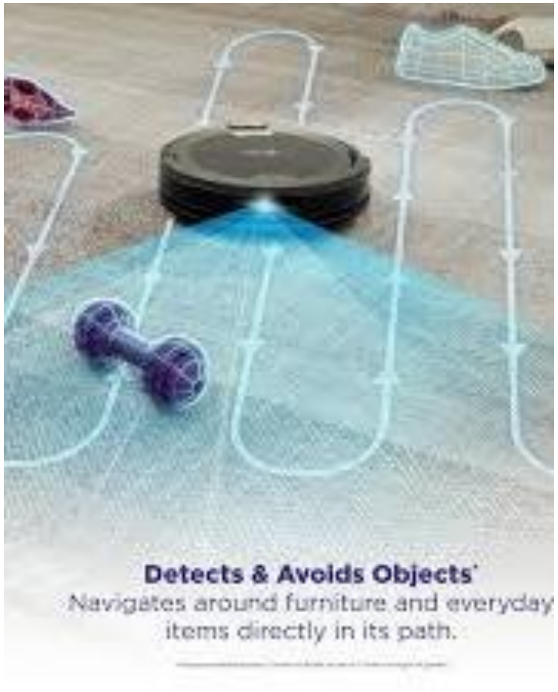
- Example: V={v1,v2,v3,v4}

  v1→v2→v3→v4→v1
  can be a Hamilton Cycle

Why It Matters?



Knight Tour Problem



Robotic Vacuum
Path finding

One Stroke Puzzle

Getting On to problem: Brute Force Algorithm

If we given a n-vertex graph G(N,E):

- We could take and Examine every Order for this n vertexes:
  for Eg. P(N)= v1,v2,v3,……,vn
- By Order we will check if v_i has edge to v_{i+1} → check $E[v_i, v_{i+1}]$
- If All answers are true at last we check if v_n connected to v_1
- If True P is a Hamilton Cycle For Graph G

$$IsLoop(P) = \bigwedge_{1}^{n-1} E(p_i, p_{i+1})\ .E(p_n, p_1)$$

- Time Complexity:
- We have n! cases.
- for each case we check n+1 Edges
- Brute Force Solution Would Be in O((n+1)!)

Getting Into Problem : Searching For Better Solution

The Question asked to find a $O(2^n n^2)$ solution.

- What we can realize from the requested Order?

We Have A graph with n vertex, with O(n^2) possible edges,

We Can guess 2^n could be related to a subsets of main graphs
Speculatively we can guess that O(2^n) can hint us to **solve the problem for Every Single Sub Graph of G.**

So we define A sub graph S ∈ G and first we **solve sub problem for S**
Then We use the answer to Solve bigger sub problems.

Subproblem Definition

If we want to Solve Problem For Every Sub Graph(Subset Of vertexes)
We need A Suitable State:

DP State:

- For erery subset of vertexes S ∈ G and v ∈ S we define DP[S][v]:

DP[S][v]

- DP[S][v] is true if there is a Hamilton path (not cycle) starting with v_1 and ending with v

- Notice every S should contain v_1 and

$$|S| \geq 1$$

First Step: Filling DP for |S| = 1

If we start with $S_0 = \{v_1\}$ and $|S| = 1$ we can surely say:

$DP[S_0][v1] = True$

Second Step: Filling DP for |S| = 2

$$DP[\{v_1,u\}][u] = E[v_1,u]$$

These Two Steps Makes Our Base Case And Now We Can

Proof By Induction : Induction Hypothesis And Rule

Induction Step: HI
Suppose We Have k which:

$\forall i \leq k, \forall S \in V$ ^ $|S| \leq i$: DP[S][v ∈S]

- DP[S][v ∈S] is known
- All of these DP[S][v] is either 0 or 1

Rule of Induction: we can now determine all DP[S'][v]
For |S'|=k+1:

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For |S'|=k+1:

We want to determine DP[S’][v]

what we know so far:

- v ∈ S’,|S’| = k+1
- Every DP[S][v] is known if $|S| \leq k$

So what we do is we use S’-v answers to determine DP in S’:

Claim: we can know if there is a Hamilton path in S’-v.

- If there is **no Hamilton path in S’-v** then there is **no Hamilton path either in S**
- If there is a some Hamilton path in S’-v then **exist some u in S’-v that there is a Hamilton path from 1 to u**

  - So if there is a Hamilton path ending with some u:
- We check if any of them has edge to v then there is Hamilton Path from 1 to v. it mean DP[S’][v] = true

We want to determine DP[S'][v]

what we know so far:

- v ∈ S', |S'| = k+1
- Every DP[S][v] is known if $|S| \leq k$

Case1:

$$\nexists u \ \in \ \mathrm{S}' - \mathrm{v}\,, \mathrm{DP}[\mathrm{S}' - \mathrm{v}][\mathrm{u}] = \mathrm{true} \ \rightarrow \mathrm{DP}[\mathrm{S}'][\mathrm{v}] = \mathrm{false}$$

Case2:

$$\exists\, u \ \in \ \mathrm{S}' - \mathrm{v}\,, \mathrm{DP}[\mathrm{S}' - \mathrm{v}][\mathrm{u}] = \mathrm{true}, \mathrm{E}[\mathrm{u}][\mathrm{v}] = \mathrm{true} \rightarrow \mathrm{DP}[\mathrm{S}'][v] = \mathrm{false}$$

*it says :

if there is any u that there is a Hamilton path from 1 to u

and u has edge to v

then there Hamilton path from 1 to v

then DP[S'][v] = true

Formal Formula For DP[S'][v]

Case1:

$\nexists u \in S' - \mathrm{v}, \mathrm{DP}[S' - \mathrm{v}][\mathrm{u}] = \mathrm{true} \rightarrow \mathrm{DP}[S'][\mathrm{v}] = \mathrm{false}$

Case2:

$\exists\, u \in S' - \mathrm{v}, \mathrm{DP}[S' - \mathrm{v}][\mathrm{u}] = \mathrm{true}, \mathrm{E}[\mathrm{u}][\mathrm{v}] = \mathrm{true} \rightarrow \mathrm{DP}[S'][v] = \mathrm{false}$

$$DP[S'][v] = \bigvee_{u \in S' - v} DP[S' - v][u] = true \mathbin{\&\&} E[u][v] = true$$

Final Step: Complete Hamitlon Path

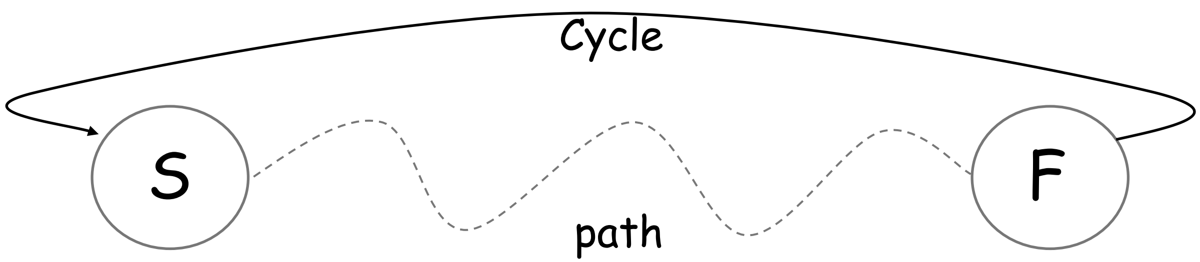
Final Condition.

- For finding Hamilton paths in then main graph G
That ends with vertex v we can check DP[V][v]
- If DP[V][v] is true we have Hamilton path from 1 to v with length N so it's a complete Hamilton Path

Final Step : From Hamilton Path to Cycle

Now how we can know if we can know Hamilton Cycles if we Know Hamilton Paths?

HC = Hamilton Cycle, HP = Hamilton Path
HC = HP from **s to f** + Single Edge from **f to s**



So HC[1..u..1] = DP[V][u] ^ E[u][1]

Final Step:Is There Hamilton Cycle?

Def. If HC = true we Have Hamilton Cycle on graph G

$$HC = \bigvee_{v \in V} DP[V][v] = true \ \&\& \ E[v][1] = true$$

Final Step:Is There Hamilton Cycle?

If HC = true we Have Hamilton Cycle on graph G Because:

We have Hamilton Way from 1 to v

Also we have edge from v to 1

So there is A Hamilton cycle like 1→………→v→1

$$HC = \bigvee_{v \in V} DP[V][v] = true \ \&\& \ E[v][1] = true$$

HC = true → we have at least one Hamilton cycle

Now What if HC = False?

Now What if HC = False?

- If HC = false it means there is no u which has both
  1_edge from u to 1
  2_hamiton path from 1 to u

If We Have A Hamilton Cycle:
1-every vertex should Have Positive Input Degree
So $D_1>0$
2-if there is a cycle with length n+1 there is a path
*It means exists vertex v that there is
HP from 1 to v and edge from v to 1

So For That “v”→ DP[V][v] = true ^ E[v][1] = true
SOO?? We Can Say Surely Then **HC=TRUE**

Two Way Condition

Has been Proved HC=True → There is a Hamilton Cycle
Also Existing of Hamilton Cycle → HC =True

Then It's Two Way condition: HC == Existing Of Hamilton Cycle

if HC=0 false there is no Hamilton Cycle

Time Complexity of algorithm

At the Worst case for Filling Every DP[S][v]
**We Check all DP[S-v][u] → O(n)

We have: (2^n case for S) * (n case for v) * (n case for u) SO→

$T(n) \in O(2^n n^2)$.
We found Algorithm in Requested Time with proof.