

Graph (overview)

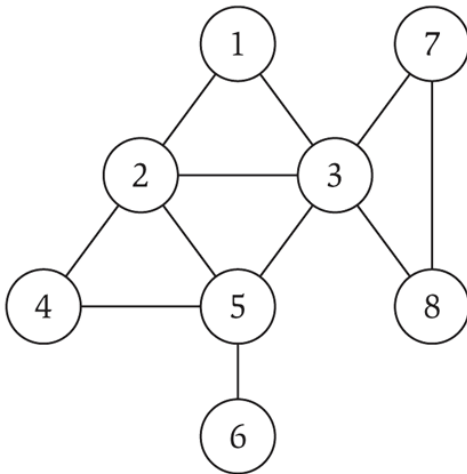
- Basic Definition and Applications
- Graph Traversal (BFS and DFS)
- Testing Bipartiteness
- Connectivity in Directed Graphs
- DAGs and Topological Ordering
- Shortest Paths in a Graph
- Negative Cycles in a Graph

Basic Definitions and Applications

Undirected Graphs

Undirected graph. $G = (V, E)$

- V = nodes.
- E = edges between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters: $n = |V|$, $m = |E|$.



$V = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$

$E = \{ 1-2, 1-3, 2-3, 2-4, 2-5, 3-5, 3-7, 3-8, 4-5, 5-6 \}$

$n = 8$

$m = 11$

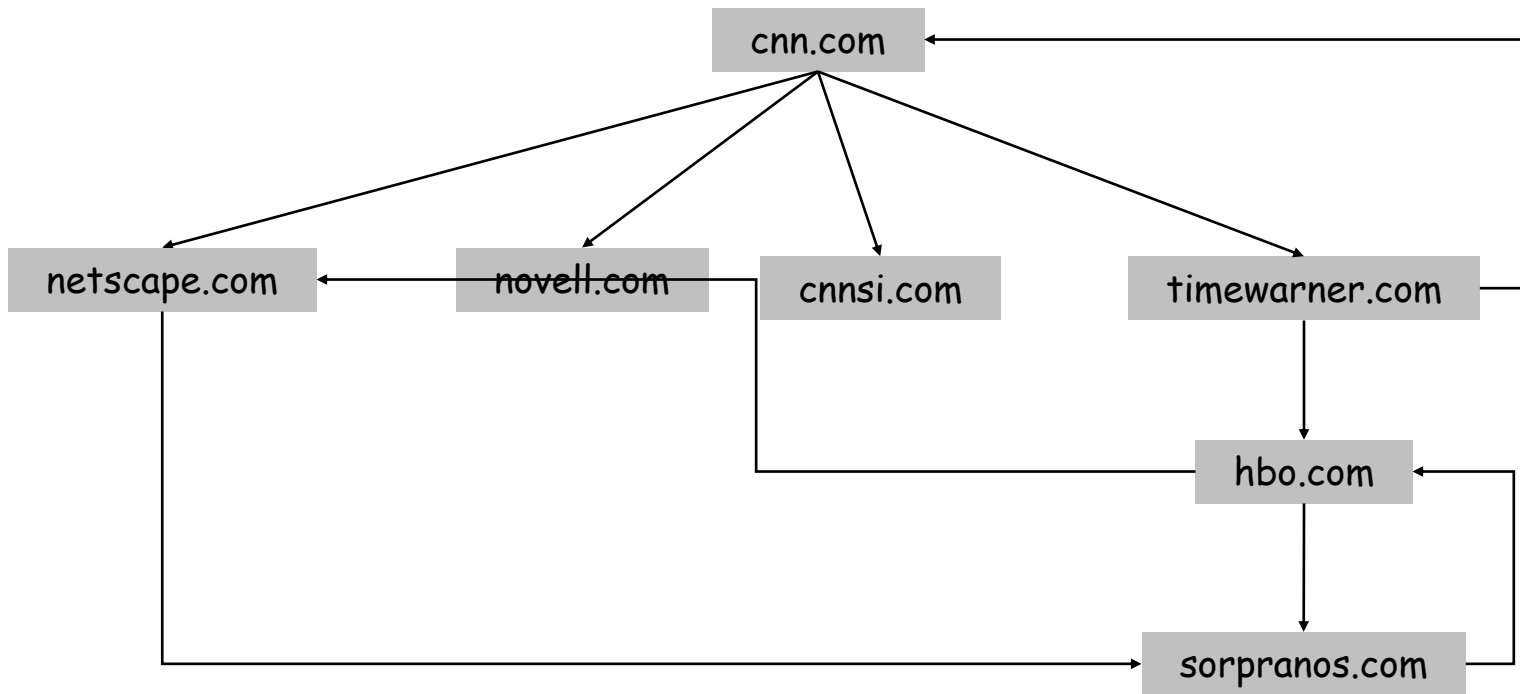
Some Graph Applications

<i>Graph</i>	<i>Nodes</i>	<i>Edges</i>
transportation	street intersections	highways
communication	computers	fiber optic cables
World Wide Web	web pages	hyperlinks
social	people	relationships
food web	species	predator-prey
software systems	functions	function calls
scheduling	tasks	precedence constraints
circuits	gates	wires

World Wide Web

Web graph.

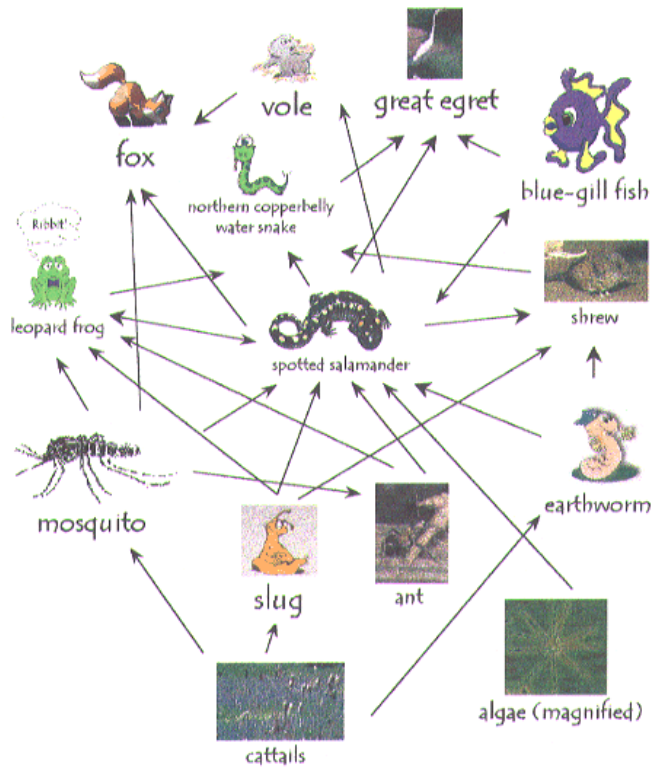
- Node: web page.
- Edge: hyperlink from one page to another.



Ecological Food Web

Food web graph.

- Node = species.
- Edge = from prey to predator.

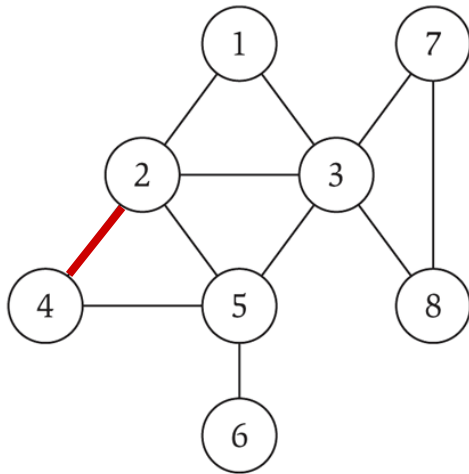


Reference: <http://www.twingroves.district96.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.gif>

Graph Representation: Adjacency Matrix

Adjacency matrix. n -by- n matrix with $A_{uv} = 1$ if (u, v) is an edge.

- Two representations of each edge.
- Space proportional to n^2 .
- Checking if (u, v) is an edge takes $\Theta(1)$ time.
- Identifying all edges takes $\Theta(n^2)$ time.



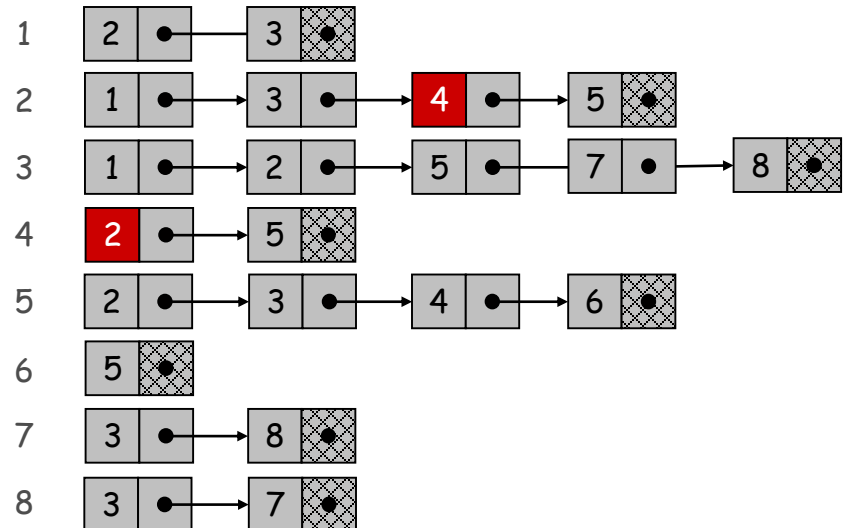
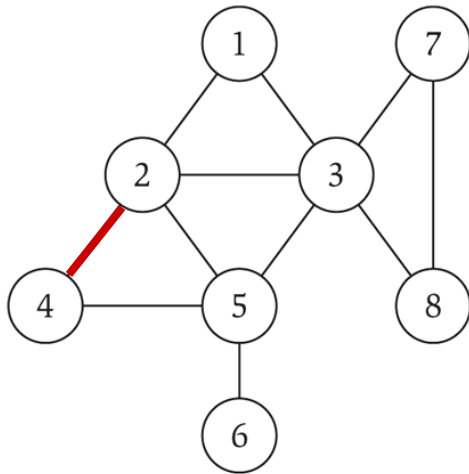
	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	1	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

Graph Representation: Adjacency List

Adjacency list. Node indexed array of lists.

- Two representations of each edge.
- Space proportional to $m + n$.
- Checking if (u, v) is an edge takes $O(\text{deg}(u))$ time.
- Identifying all edges takes $\Theta(m + n)$ time.

degree = number of neighbors of u

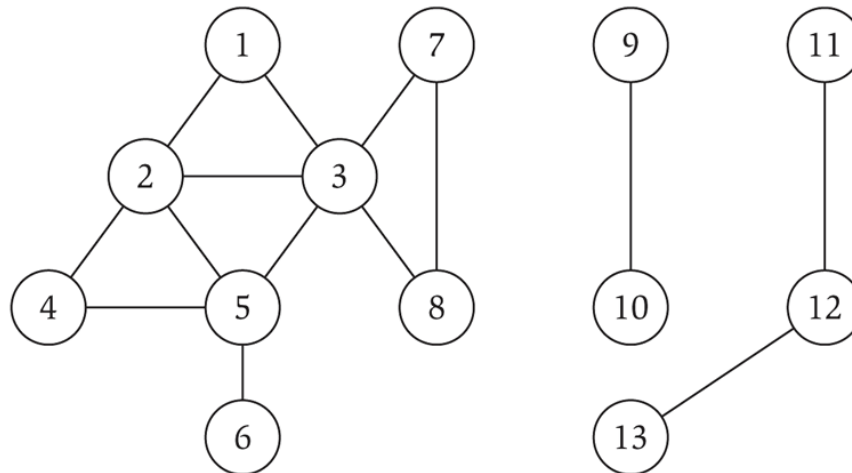


Paths and Connectivity

Def. A **path** in an undirected graph $G = (V, E)$ is a sequence P of nodes $v_1, v_2, \dots, v_{k-1}, v_k$ with the property that each consecutive pair v_i, v_{i+1} is joined by an edge in E .

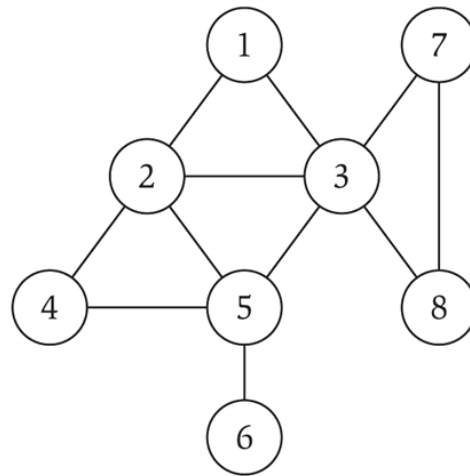
Def. A path is **simple** if all nodes are distinct.

Def. An undirected graph is **connected** if for every pair of nodes u and v , there is a path between u and v .



Cycles

Def. A **cycle** is a path $v_1, v_2, \dots, v_{k-1}, v_k$ in which $v_1 = v_k$, $k > 2$, and the first $k-1$ nodes are all distinct.



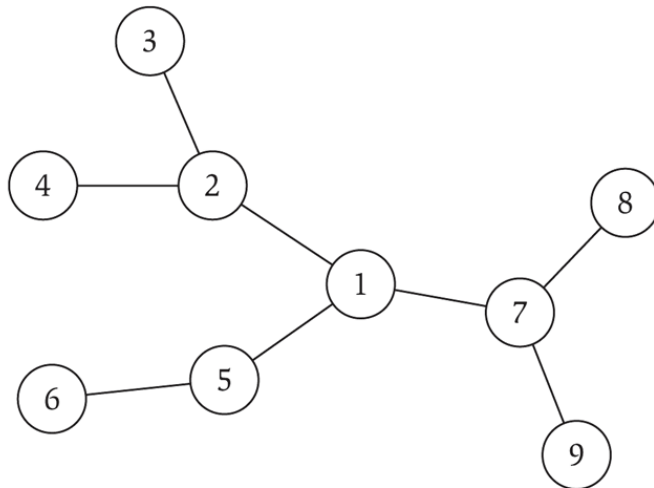
cycle $C = 1-2-4-5-3-1$

Trees

Def. An undirected graph is a **tree** if it is connected and does not contain a cycle.

Theorem. Let G be an undirected graph on n nodes. Any two of the following statements imply the third.

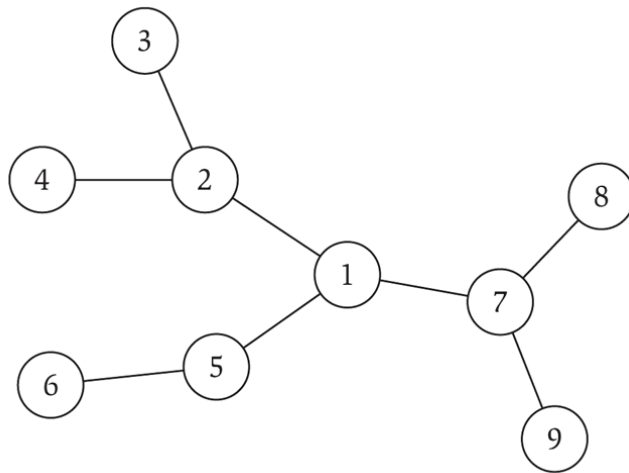
- G is connected.
- G does not contain a cycle.
- G has $n-1$ edges.



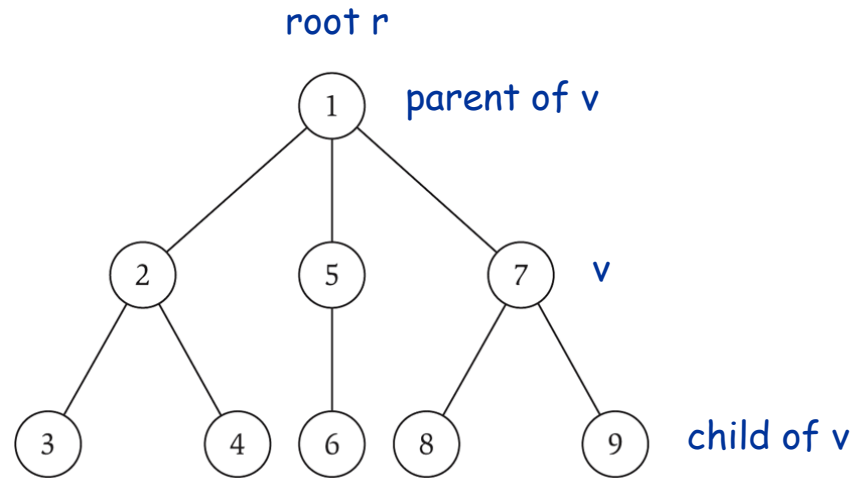
Rooted Trees

Rooted tree. Given a tree T , choose a root node r and orient each edge away from r .

Importance. Models hierarchical structure.



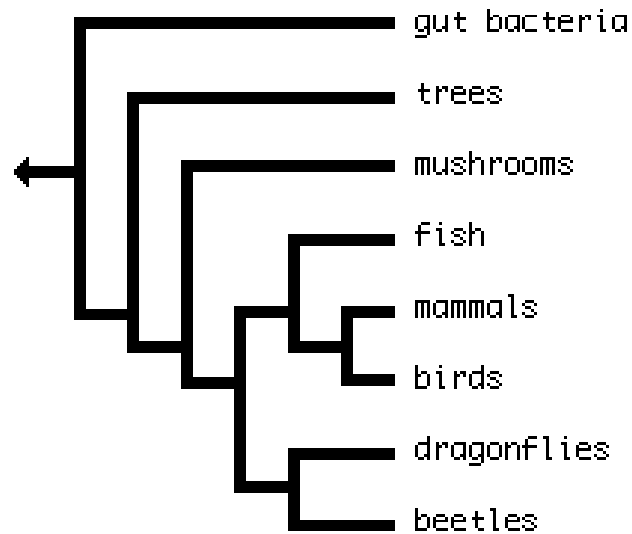
a tree



the same tree, rooted at 1

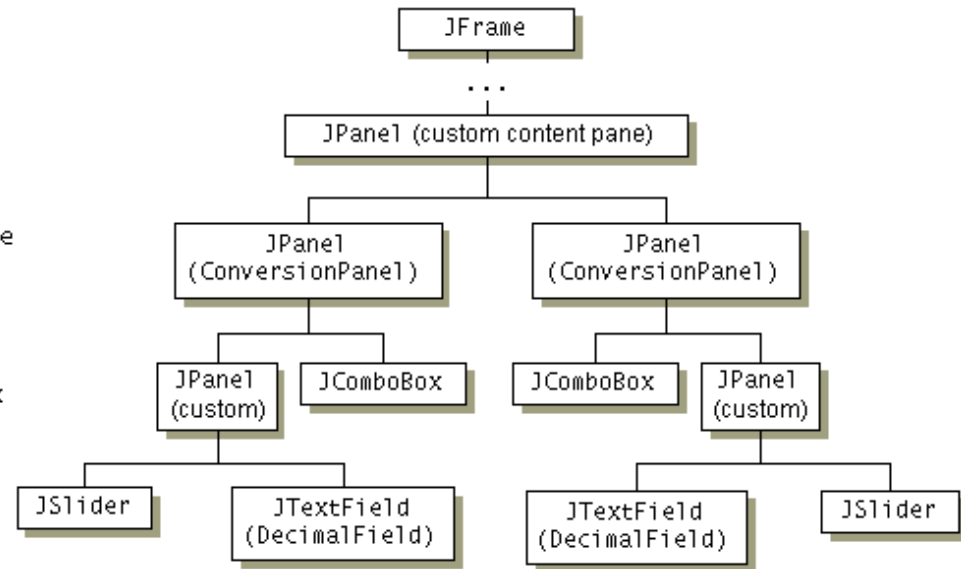
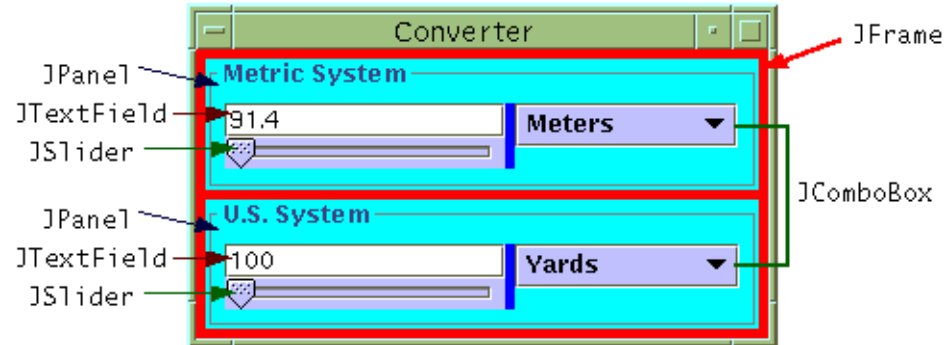
Phylogeny Trees

Phylogeny trees. Describe evolutionary history of species.



GUI Containment Hierarchy

GUI containment hierarchy. Describe organization of GUI widgets.



Reference: <http://java.sun.com/docs/books/tutorial/uiswing/overview/anatomy.html>

Graph Traversal

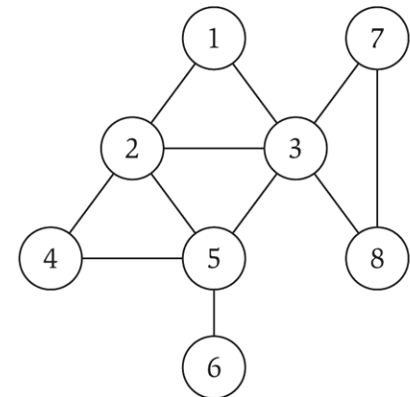
Connectivity

s-t connectivity problem. Given two nodes s and t , is there a path between s and t ?

s-t shortest path problem. Given two nodes s and t , what is the length of the shortest path between s and t ?

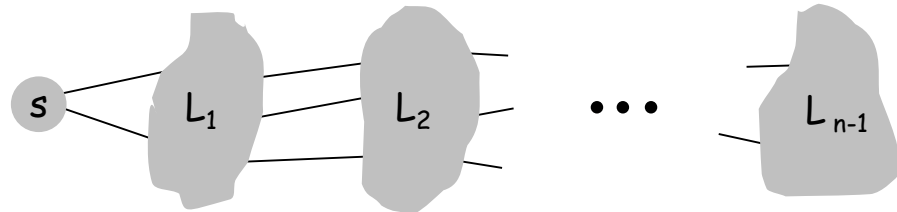
Applications.

- Maze traversal.
- Erdos number.
- Fewest number of hops in a communication network.



Breadth First Search

BFS intuition. Explore outward from s in all possible directions, adding nodes one "layer" at a time.



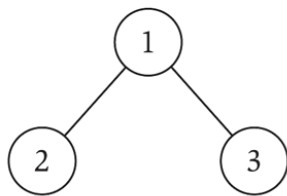
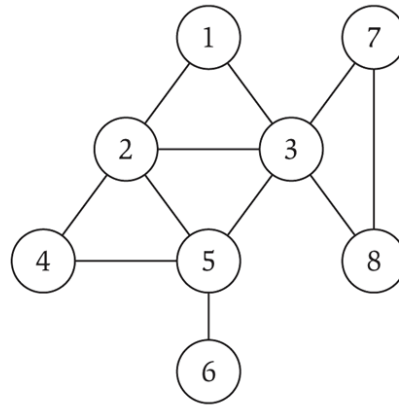
BFS algorithm.

- $L_0 = \{ s \}$.
- $L_1 =$ all neighbors of L_0 .
- $L_2 =$ all nodes that do not belong to L_0 or L_1 , and that have an edge to a node in L_1 .
- $L_{i+1} =$ all nodes that do not belong to an earlier layer, and that have an edge to a node in L_i .

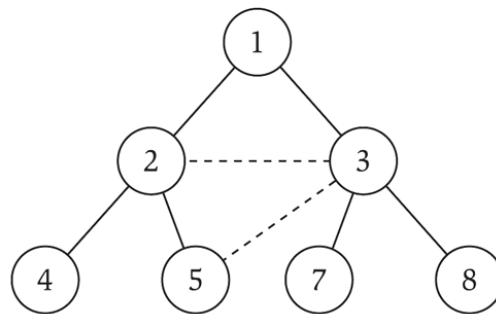
Theorem. For each i , L_i consists of all nodes at distance exactly i from s . There is a path from s to t iff t appears in some layer.

Breadth First Search

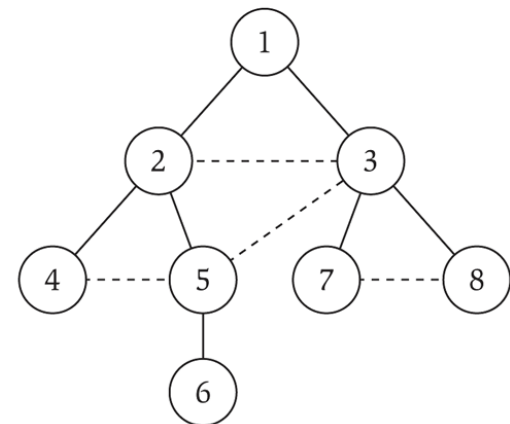
Property. Let T be a BFS tree of $G = (V, E)$, and let (x, y) be an edge of G . Then the level of x and y differ by at most 1.



(a)



(b)



(c)

L_0

L_1

L_2

L_3

Breadth First Search: Analysis

Theorem. The above implementation of BFS runs in $O(m + n)$ time if the graph is given by its adjacency representation.

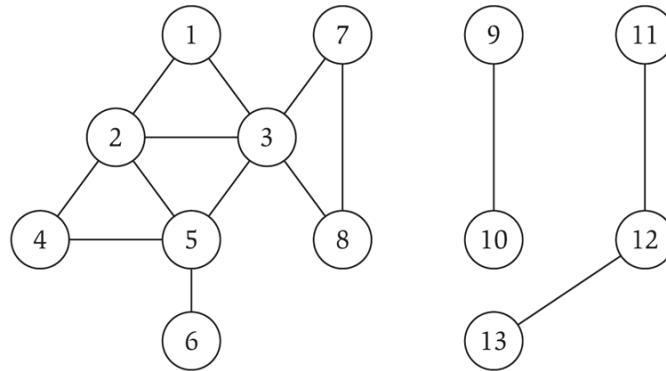
Pf.

- Easy to prove $O(n^2)$ running time:
 - at most n lists $L[i]$
 - each node occurs on at most one list; for loop runs $\leq n$ times
 - when we consider node u , there are $\leq n$ incident edges (u, v) , and we spend $O(1)$ processing each edge
- Actually runs in $O(m + n)$ time:
 - when we consider node u , there are $\text{deg}(u)$ incident edges (u, v)
 - total time processing edges is $\sum_{u \in V} \text{deg}(u) = 2m$ ▪

↑
each edge (u, v) is counted exactly twice
in sum: once in $\text{deg}(u)$ and once in $\text{deg}(v)$

Connected Component

Connected component. Find all nodes reachable from s .



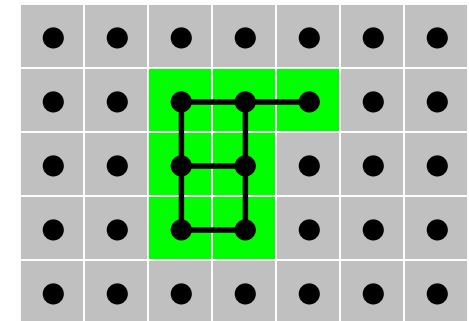
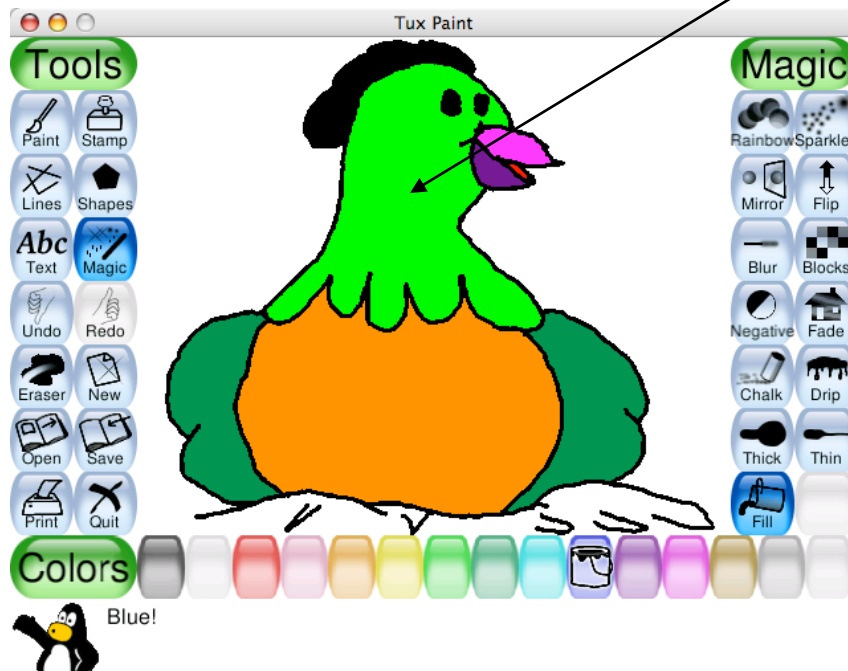
Connected component containing node 1 = { 1, 2, 3, 4, 5, 6, 7, 8 }.

Flood Fill

Flood fill. Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue.

- Node: pixel.
- Edge: two neighboring lime pixels.
- Blob: connected component of lime pixels.

recolor lime green blob to blue

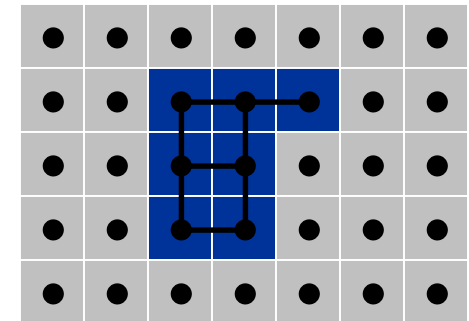


Flood Fill

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- Node: pixel.
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recolor lime green blob to blue



Connected Component

Connected component. Find all nodes reachable from s .

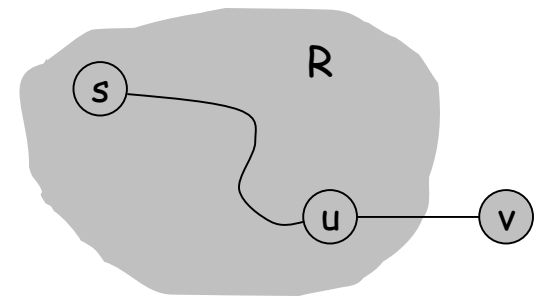
R will consist of nodes to which s has a path

Initially $R = \{s\}$

While there is an edge (u, v) where $u \in R$ and $v \notin R$

 Add v to R

Endwhile



it's safe to add v

Theorem. Upon termination, R is the connected component containing s .

- BFS = explore in order of distance from s .
- DFS = explore in a different way.

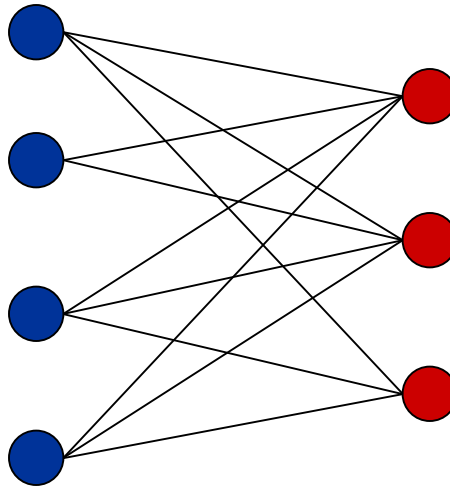
Testing Bipartiteness

Bipartite Graphs

Def. An undirected graph $G = (V, E)$ is **bipartite** if the nodes can be colored red or blue such that every edge has one red and one blue end.

Applications.

- Stable marriage: men = red, women = blue.
- Scheduling: machines = red, jobs = blue.

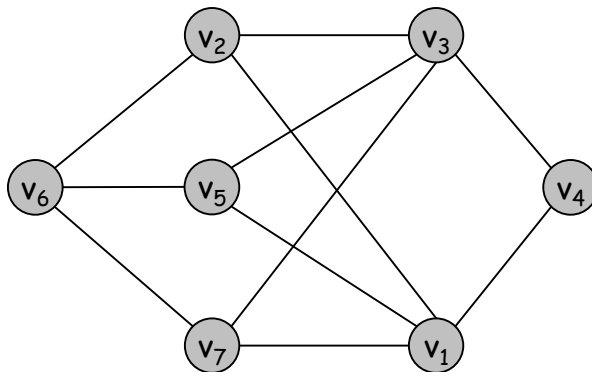


a bipartite graph

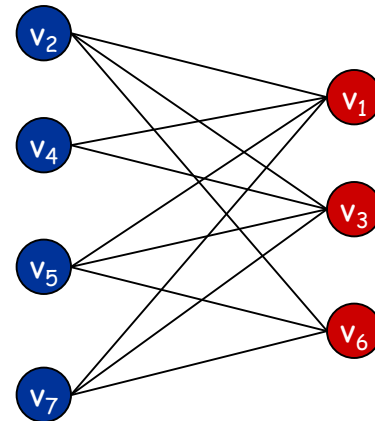
Testing Bipartiteness

Testing bipartiteness. Given a graph G , is it bipartite?

- Many graph problems become:
 - easier if the underlying graph is bipartite (matching)
 - tractable if the underlying graph is bipartite (independent set)
- Before attempting to design an algorithm, we need to understand structure of bipartite graphs.



a bipartite graph G

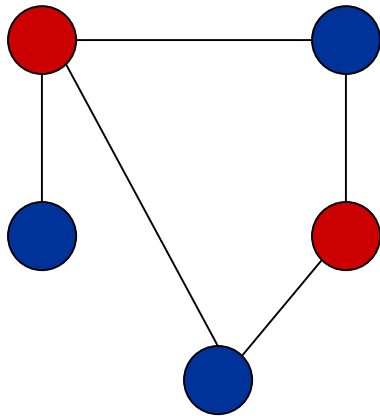


another drawing of G

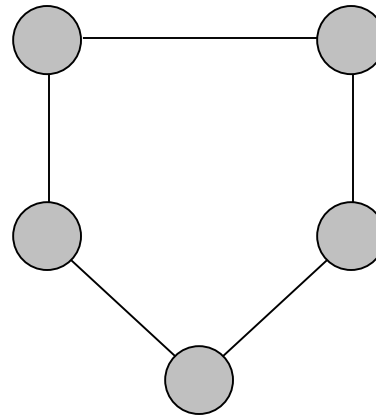
An Obstruction to Bipartiteness

Lemma. If a graph G is bipartite, it cannot contain an odd length cycle.

Pf. Not possible to 2-color the odd cycle, let alone G .



bipartite
(2-colorable)

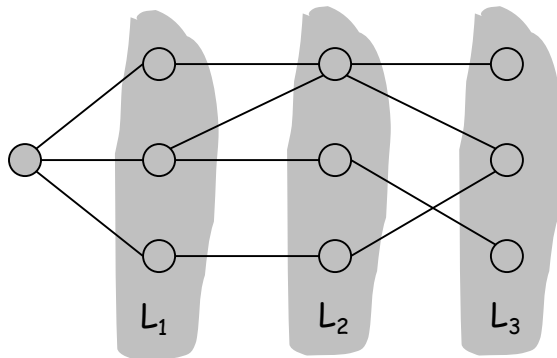


not bipartite
(not 2-colorable)

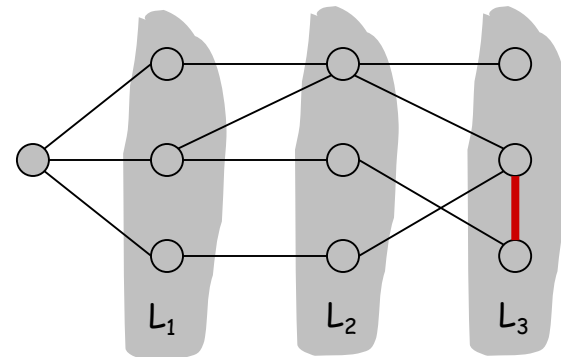
Bipartite Graphs

Lemma. Let G be a connected graph, and let L_0, \dots, L_k be the layers produced by BFS starting at node s . Exactly one of the following holds.

- (i) No edge of G joins two nodes of the same layer, and G is bipartite.
- (ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).



Case (i)



Case (ii)

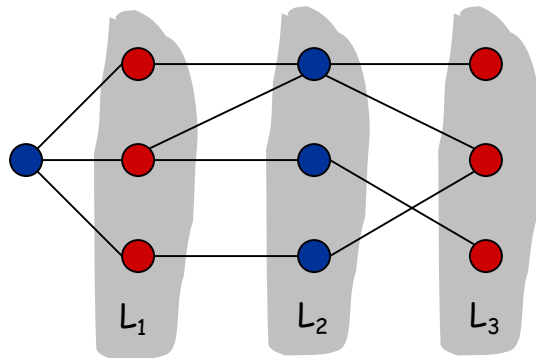
Bipartite Graphs

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- (i) No edge of G joins two nodes of the same layer, and G is bipartite.
- (ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Pf. (i)

- Suppose no edge joins two nodes in adjacent layers.
- By previous lemma, this implies all edges join nodes on same level.
- Bipartition: red = nodes on odd levels, blue = nodes on even levels.



Case (i)

Bipartite Graphs

Lemma. Let G be a connected graph, and let L_0, \dots, L_k be the layers produced by BFS starting at node s . Exactly one of the following holds.

- (i) No edge of G joins two nodes of the same layer, and G is bipartite.
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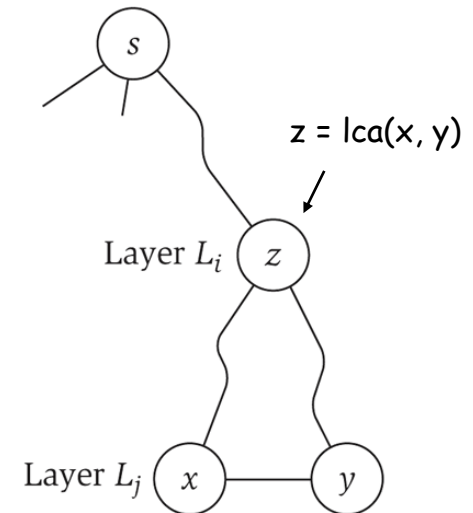
Pf. (ii)

- Suppose (x, y) is an edge with x, y in same level L_j .
- Let $z = \text{lca}(x, y) =$ lowest common ancestor.
- Let L_i be level containing z .
- Consider cycle that takes edge from x to y , then path from y to z , then path from z to x .
- Its length is $1 + \underbrace{(j-i)}_{\text{path from } y \text{ to } z} + \underbrace{(j-i)}_{\text{path from } z \text{ to } x}$, which is odd. ▪

$\underbrace{\hspace{1.5cm}}$
 (x, y)

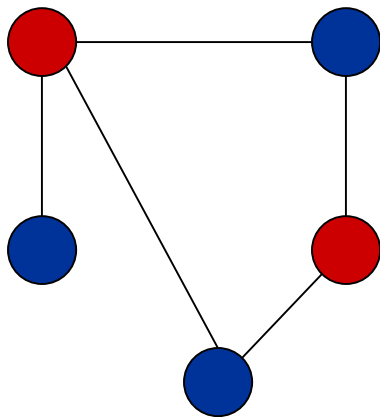
$\underbrace{\hspace{1.5cm}}$
 path from y to z

$\underbrace{\hspace{1.5cm}}$
 path from z to x

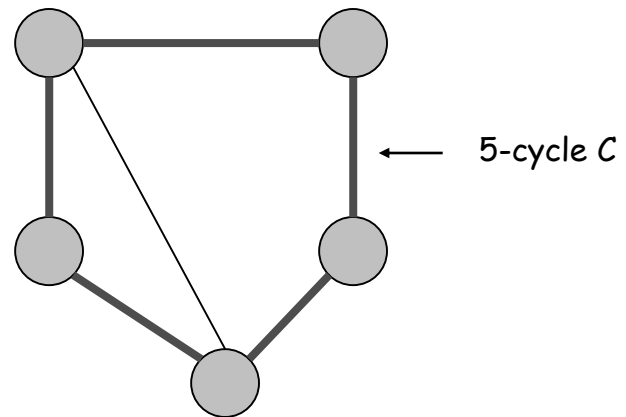


Obstruction to Bipartiteness

Corollary. A graph G is bipartite iff it contains no odd length cycle.



bipartite
(2-colorable)



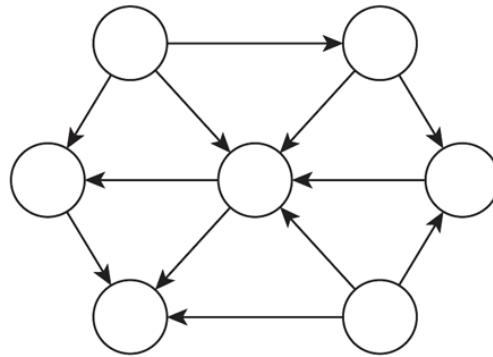
not bipartite
(not 2-colorable)

Connectivity in Directed Graphs

Directed Graphs

Directed graph. $G = (V, E)$

- Edge (u, v) goes from node u to node v .



Ex. Web graph - hyperlink points from one web page to another.

- Directedness of graph is crucial.
- Modern web search engines exploit hyperlink structure to rank web pages by importance.

Graph Search

Directed reachability. Given a node s , find all nodes reachable from s .

Directed s - t shortest path problem. Given two nodes s and t , what is the length of the shortest path between s and t ?

Graph search. BFS extends naturally to directed graphs.

Web crawler. Start from web page s . Find all web pages linked from s , either directly or indirectly.

Strong Connectivity

Def. Node u and v are **mutually reachable** if there is a path from u to v and also a path from v to u .

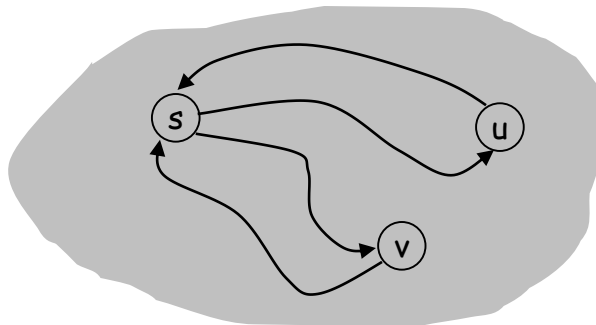
Def. A graph is **strongly connected** if every pair of nodes is mutually reachable.

Lemma. Let s be any node. G is strongly connected iff every node is reachable from s , and s is reachable from every node.

Pf. \Rightarrow Follows from definition.

Pf. \Leftarrow Path from u to v : concatenate u - s path with s - v path.

Path from v to u : concatenate v - s path with s - u path. ■



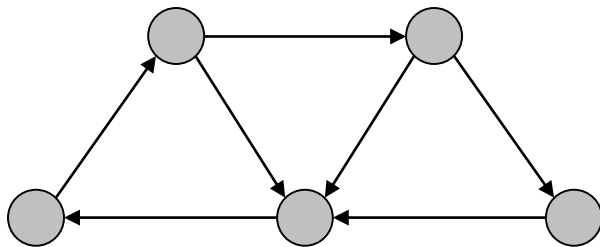
↖
ok if paths overlap

Strong Connectivity: Algorithm

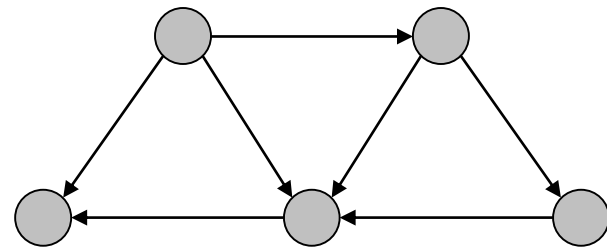
Theorem. Can determine if G is strongly connected in $O(m + n)$ time.

Pf.

- Pick any node s .
- Run BFS from s in G .
- Run BFS from s in G^{rev} . reverse orientation of every edge in G
- Return true iff all nodes reached in both BFS executions.
- Correctness follows immediately from previous lemma. ▪



strongly connected



not strongly connected

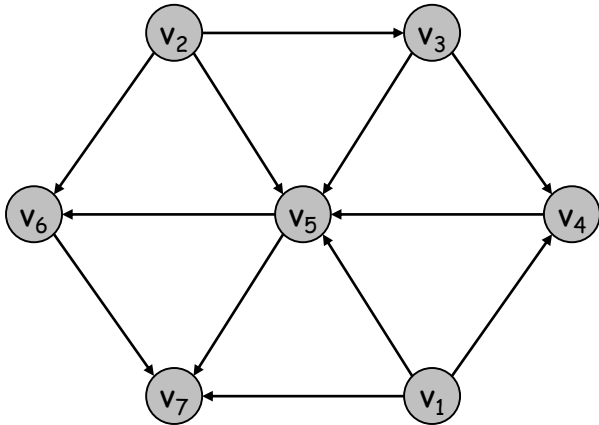
DAGs and Topological Ordering

Directed Acyclic Graphs

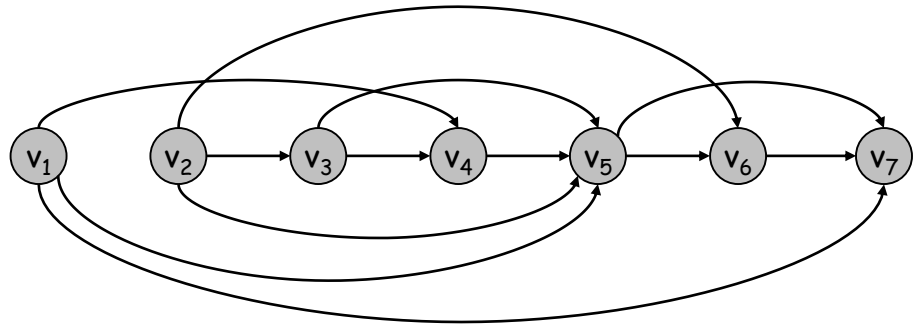
Def. An **DAG** is a directed graph that contains no directed cycles.

Ex. Precedence constraints: edge (v_i, v_j) means v_i must precede v_j .

Def. A **topological order** of a directed graph $G = (V, E)$ is an ordering of its nodes as v_1, v_2, \dots, v_n so that for every edge (v_i, v_j) we have $i < j$.



a DAG



a topological ordering

Precedence Constraints

Precedence constraints. Edge (v_i, v_j) means task v_i must occur before v_j .

Applications.

- Course prerequisite graph: course v_i must be taken before v_j .
- Compilation: module v_i must be compiled before v_j . Pipeline of computing jobs: output of job v_i needed to determine input of job v_j .

Directed Acyclic Graphs

Lemma. If G has a topological order, then G is a DAG.

Q. Does every DAG have a topological ordering?

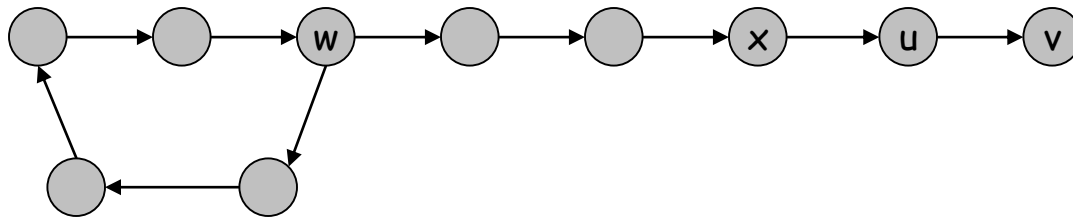
Q. If so, how do we compute one?

Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)

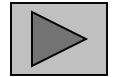
- Suppose that G is a DAG and every node has at least one incoming edge. Let's see what happens.
- Pick any node v , and begin following edges backward from v . Since v has at least one incoming edge (u, v) we can walk backward to u .
- Then, since u has at least one incoming edge (x, u) , we can walk backward to x .
- Repeat until we visit a node, say w , twice.
- Let C denote the sequence of nodes encountered between successive visits to w . C is a cycle. ▪



Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)



- Base case: true if $n = 1$.
- Given DAG on $n > 1$ nodes, find a node v with no incoming edges.
- $G - \{v\}$ is a DAG, since deleting v cannot create cycles.
- By inductive hypothesis, $G - \{v\}$ has a topological ordering.
- Place v first in topological ordering; then append nodes of $G - \{v\}$
- in topological order. This is valid since v has no incoming edges. ▪

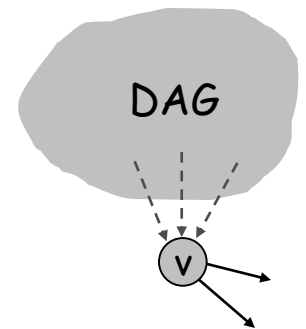
To compute a topological ordering of G :

Find a node v with no incoming edges and order it first

Delete v from G

Recursively compute a topological ordering of $G - \{v\}$

and append this order after v



Topological Sorting Algorithm: Running Time

Theorem. Algorithm finds a topological order in $O(m + n)$ time.

Pf.

- Maintain the following information:
 - $\text{count}[w]$ = remaining number of incoming edges
 - S = set of remaining nodes with no incoming edges
- Initialization: $O(m + n)$ via single scan through graph.
- Update: to delete v
 - remove v from S
 - decrement $\text{count}[w]$ for all edges from v to w , and add w to S if $\text{count}[w]$ hits 0
 - this is $O(1)$ per edge ▪

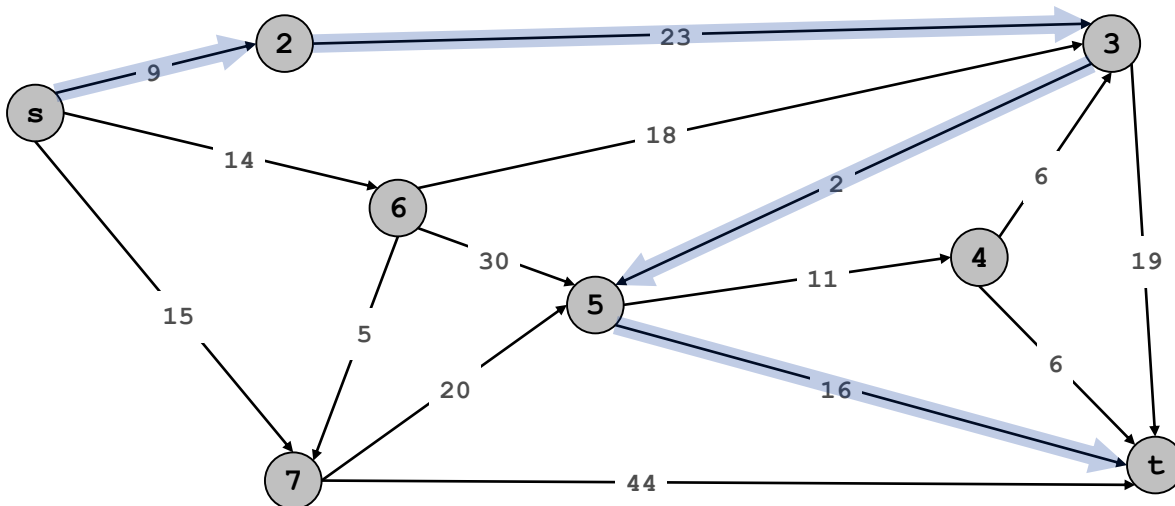
Shortest Path Problem

Shortest path network.

- Directed graph $G = (V, E)$.
- Source s , destination t .
- Length ℓ_e = length of edge e .

Shortest path problem: find shortest directed path from s to t .

↑
cost of path = sum of edge costs in path



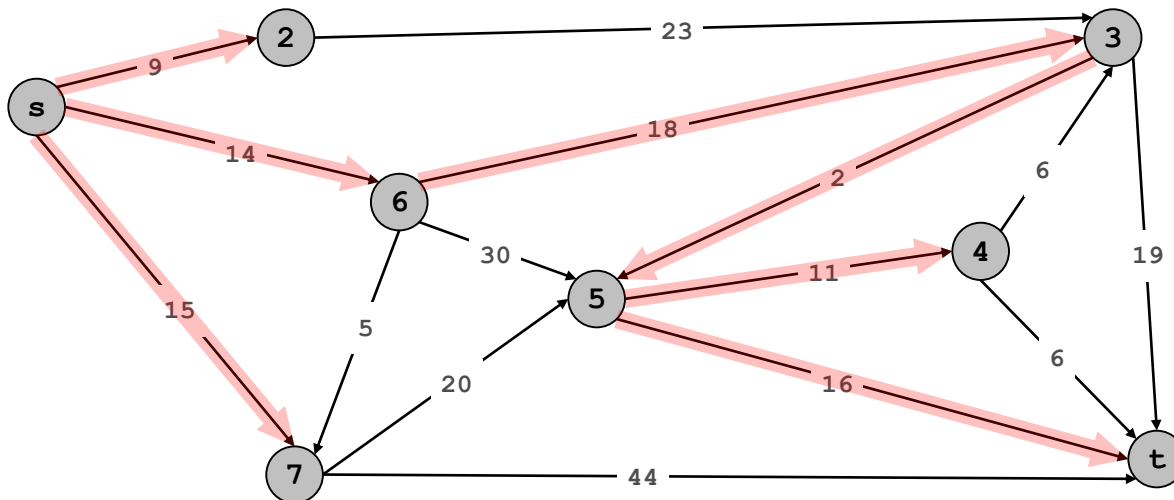
Cost of path $s-2-3-5-t$
= $9 + 23 + 2 + 16$
= 50.

Shortest Path Tree

Shortest Path Tree (SPT).

- A rooted tree with root s .
- $P(v)=u$ where s, \dots, u, v is a shortest path from s to v .
- If there are more than one shortest path from s to v , select one and define the parent of v based on it.

Single Source Shortest Path problem (SSSP): for given s , compute SPT.



Dijkstra's Algorithm

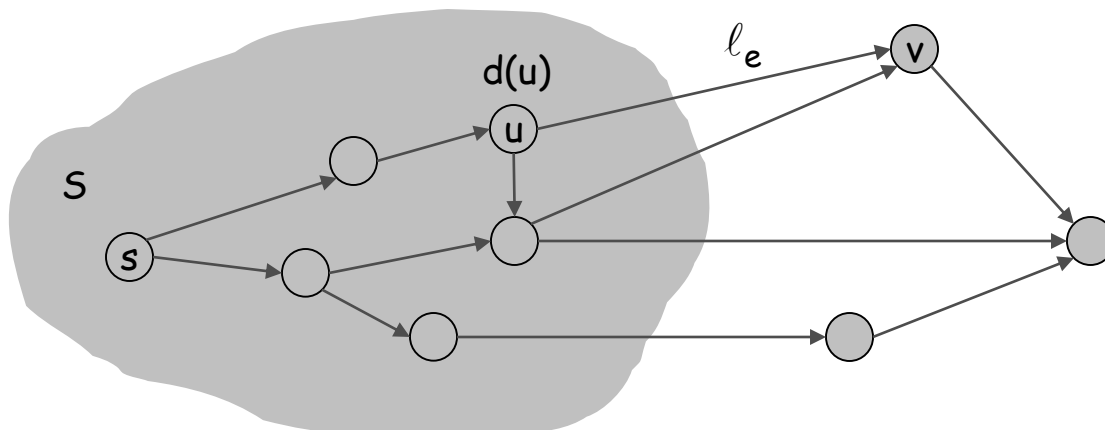
Dijkstra's algorithm. Compute SPT step by step

- Maintain a set of **explored nodes** S for which we have determined the shortest path distance $d(u)$ from s to u .
- Initialize $S = \{s\}$, $d(s) = 0$.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e=(u,v): u \in S} d(u) + \ell_e,$$

add v to S , and set $d(v) = \pi(v)$.

shortest path to some u in explored part, followed by a single edge (u, v)



Dijkstra's Algorithm

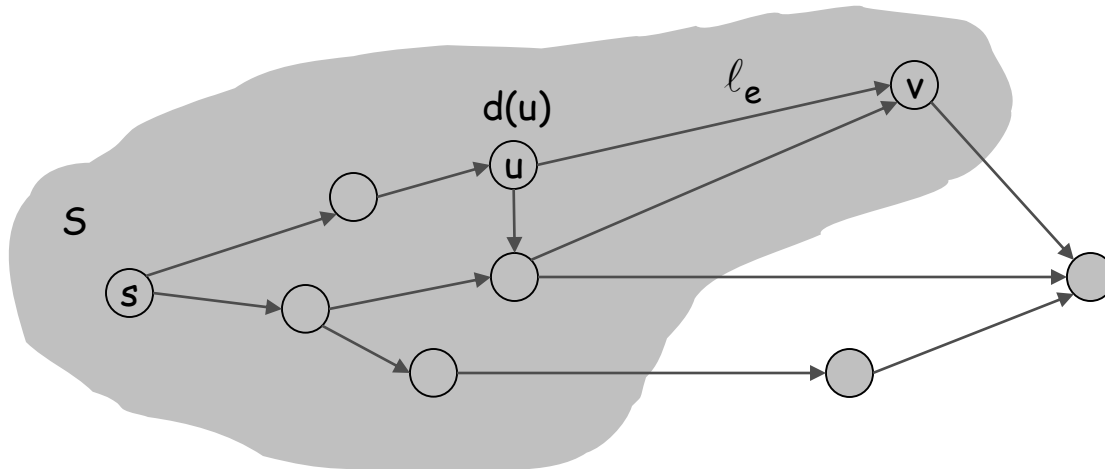
Dijkstra's algorithm.

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Dijkstra's Algorithm: Proof of Correctness

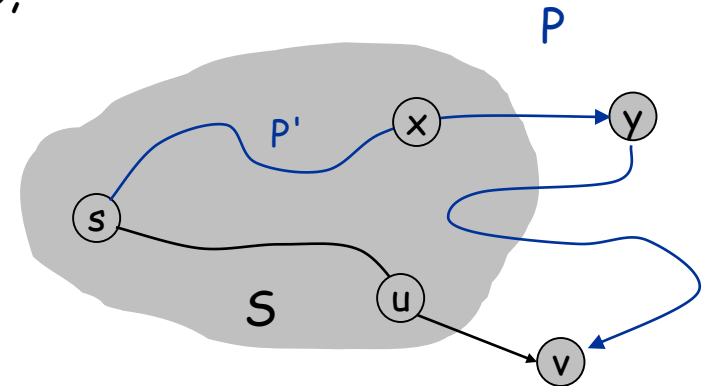
Invariant. For each node $u \in S$, $d(u)$ is the length of the shortest s - u path.

Pf. (by induction on $|S|$)

Base case: $|S| = 1$ is trivial.

Inductive hypothesis: Assume true for $|S| = k \geq 1$.

- Let v be next node added to S , and let u - v be the chosen edge.
- The shortest s - u path plus (u, v) is an s - v path of length $\pi(v)$.
- Consider any s - v path P . We'll see that it's no shorter than $\pi(v)$.
- Let x - y be the first edge in P that leaves S , and let P' be the subpath to x .
- P is already too long as soon as it leaves S .



$$\begin{array}{ccccccc}
 l(P) & \geq & l(P') + l(x, y) & \geq & d(x) + l(x, y) & \geq & \pi(y) \geq \pi(v) \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \text{nonnegative} & & \text{inductive} & & \text{defn of } \pi(y) & & \text{Dijkstra chose } v \\
 \text{weights} & & \text{hypothesis} & & & & \text{instead of } y
 \end{array}$$

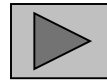
Dijkstra's Algorithm: Implementation

For each unexplored node, explicitly maintain $\pi(v) = \min_{e=(u,v): u \in S} d(u) + \ell_e$.

- Next node to explore = node with minimum $\pi(v)$.
- When exploring v , for each incident edge $e = (v, w)$, update

$$\pi(w) = \min \{ \pi(w), \pi(v) + \ell_e \}.$$

Efficient implementation. Maintain a priority queue of unexplored nodes, prioritized by $\pi(v)$.

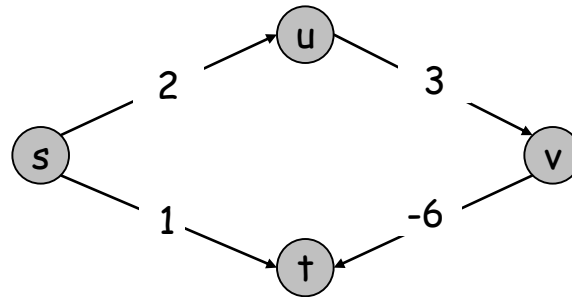


PQ Operation	Dijkstra	Array	Binary heap	d-way Heap	Fib heap †
Insert	n	n	$\log n$	$d \log_d n$	1
ExtractMin	n	n	$\log n$	$d \log_d n$	$\log n$
ChangeKey	m	1	$\log n$	$\log_d n$	1
IsEmpty	n	1	1	1	1
Total		n^2	$m \log n$	$m \log_{m/n} n$	$m + n \log n$

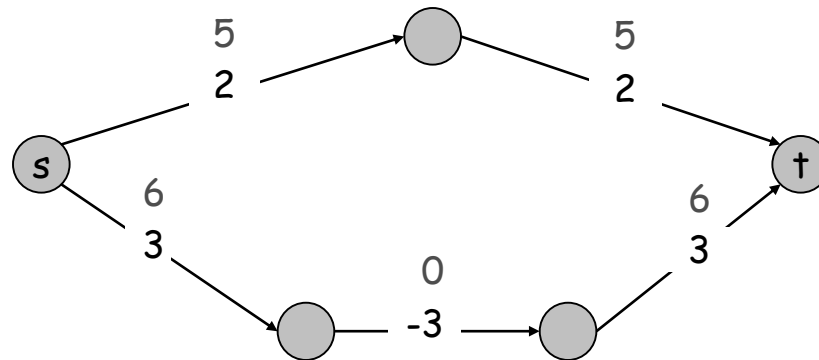
† Individual ops are amortized bounds

Shortest Path: Negative Weights

Dijkstra. Can fail if negative edge costs.

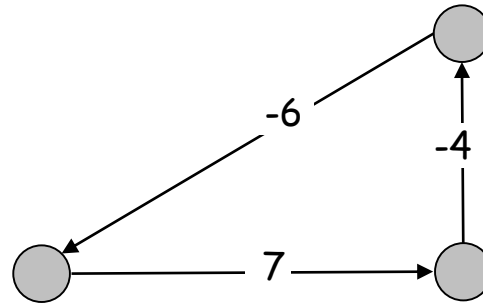


Re-weighting. Adding a constant to every edge weight can fail.

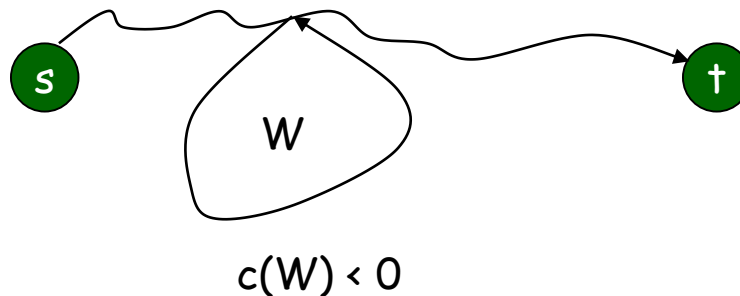


Shortest Paths: Negative Cost Cycles

Negative cost cycle.



Observation. If some path from s to t contains a negative cost cycle, there does not exist a shortest s - t path; otherwise, there exists one that is simple.



Shortest Paths: Dynamic Programming

Def. $OPT(i, v)$ = length of shortest v - t path P using at most i edges.

- Case 1: P uses at most $i-1$ edges.
 - $OPT(i, v) = OPT(i-1, v)$
- Case 2: P uses exactly i edges.
 - if (v, w) is first edge, then OPT uses (v, w) , and then selects best w - t path using at most $i-1$ edges

$$OPT(i, v) = \begin{cases} 0 & \text{if } i=0 \\ \min \left\{ OPT(i-1, v), \min_{(v, w) \in E} \{ OPT(i-1, w) + c_{vw} \} \right\} & \text{otherwise} \end{cases}$$

Remark. By previous observation, if no negative cycles, then $OPT(n-1, v)$ = length of shortest v - t path.

Shortest Paths: Implementation

```
Shortest-Path(G, t) {  
  foreach node v ∈ V  
    M[0, v] ← ∞  
  M[0, t] ← 0  
  
  for i = 1 to n-1  
    foreach node v ∈ V  
      M[i, v] ← M[i-1, v]  
      foreach edge (v, w) ∈ E  
        M[i, v] ← min { M[i, v], M[i-1, w] + cvw }  
}
```

Analysis. $\Theta(mn)$ time, $\Theta(n^2)$ space.

Finding the shortest paths. Maintain a "successor" for each table entry.

Shortest Paths: Practical Improvements

Practical improvements.

- Maintain only one array $M[v]$ = shortest v-t path that we have found so far.
- No need to check edges of the form (v, w) unless $M[w]$ changed in previous iteration.

Theorem. Throughout the algorithm, $M[v]$ is length of some v-t path, and after i rounds of updates, the value $M[v]$ is no larger than the length of shortest v-t path using $\leq i$ edges.

Overall impact.

- Memory: $O(m + n)$.
- Running time: $O(mn)$ worst case, but substantially faster in practice.

Bellman-Ford: Efficient Implementation

```
Push-Based-Shortest-Path(G, s, t) {
  foreach node v ∈ V {
    M[v] ← ∞
    successor[v] ← φ
  }

  M[t] = 0
  for i = 1 to n-1 {
    foreach node w ∈ V {
      if (M[w] has been updated in previous iteration) {
        foreach node v such that (v, w) ∈ E {
          if (M[v] > M[w] + cvw) {
            M[v] ← M[w] + cvw
            successor[v] ← w
          }
        }
      }
    }
    If no M[w] value changed in iteration i, stop.
  }
}
```

Negative Cycles in a Graph

Detecting Negative Cycles

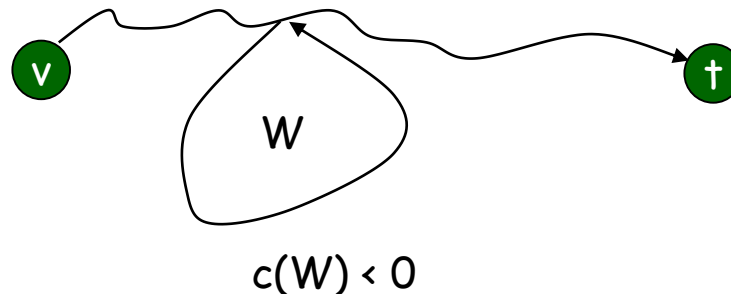
Lemma. If $\text{OPT}(n,v) = \text{OPT}(n-1,v)$ for all v , then no negative cycles.

Pf. Bellman-Ford algorithm.

Lemma. If $\text{OPT}(n,v) < \text{OPT}(n-1,v)$ for some node v , then (any) shortest path from v to t contains a cycle W . Moreover W has negative cost.

Pf. (by contradiction)

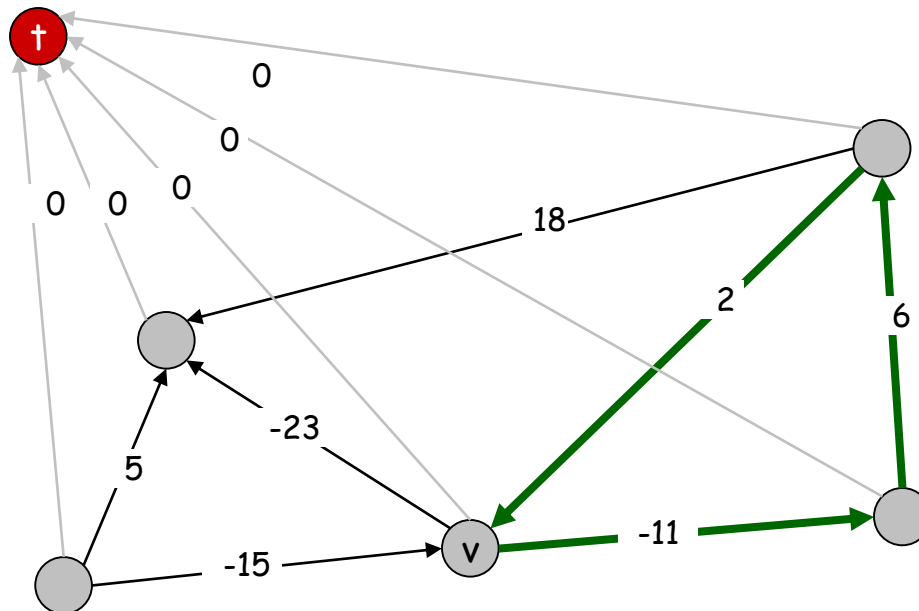
- Since $\text{OPT}(n,v) < \text{OPT}(n-1,v)$, we know P has exactly n edges.
- By pigeonhole principle, P must contain a directed cycle W .
- Deleting W yields a v - t path with $< n$ edges $\Rightarrow W$ has negative cost.



Detecting Negative Cycles

Theorem. Can detect negative cost cycle in $O(mn)$ time.

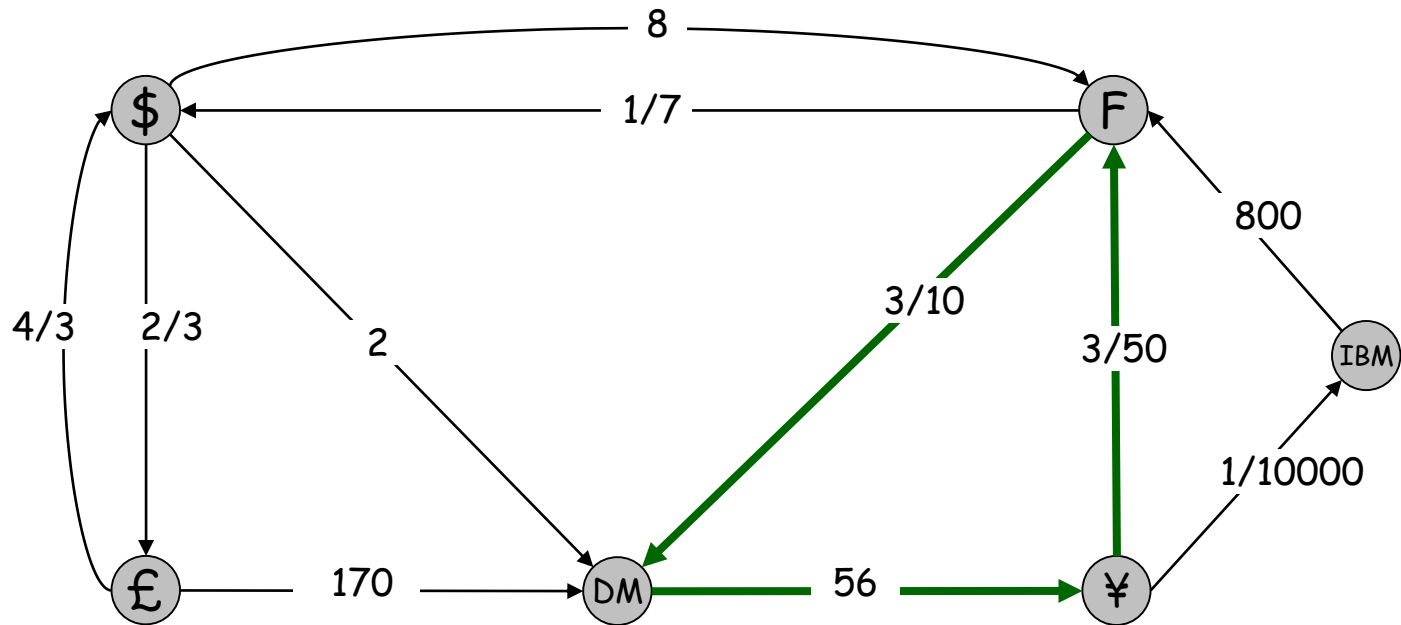
- Add new node t and connect all nodes to t with 0-cost edge.
- Check if $OPT(n, v) = OPT(n-1, v)$ for all nodes v .
 - if yes, then no negative cycles
 - if no, then extract cycle from shortest path from v to t



Detecting Negative Cycles: Application

Currency conversion. Given n currencies and exchange rates between pairs of currencies, is there an arbitrage opportunity?

Remark. Fastest algorithm very valuable!



Detecting Negative Cycles: Summary

Bellman-Ford. $O(mn)$ time, $O(m + n)$ space.

- Run Bellman-Ford for n iterations (instead of $n-1$).
- Upon termination, Bellman-Ford successor variables trace a negative cycle if one exists.
- See p. 304 for improved version and early termination rule.

References

References

- Sections 3.1-2, 3.4-6, 4.4, 6.8 and 6.10 of the text book "algorithm design" by Jon Kleinberg and Eva Tardos
- The original slides were prepared by Kevin Wayne. The slides are distributed by Pearson Addison-Wesley.