Linear Programming

- A Refreshing Problem
- Standard Forms
- Simplex Algorithm
- Duality
- Modeling Problems by LP

Linear programming

Linear programming. Optimize a linear function subject to linear inequalities.

$$
\begin{aligned}
\texttt{(P)}\ \ \texttt{max}\ & \textstyle\sum_{j=1}^{n} c_j x_j \\
\texttt{s.t.}\ & \textstyle\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad 1 \le i \le m \\
& x_j \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\texttt{(P)}\ \ \texttt{max}\ & c^T x \\
\texttt{s.t.}\ & Ax \le b \\
& x \ge 0
\end{aligned}
$$

Linear programming

Linear programming. Optimize a linear function subject to linear inequalities.

Model problems by LP: shortest path, max flow, assignment problem, matching, multicommodity flow, MST, ...

Why significant?

- Design poly-time algorithms.
- Design approximation algorithms.

Ranked among most important scientific advances of 20th century.

A Refreshing Problem

Product Problem

Small factory produces products X and Y.

- Production limited by resources: A, B, C.
- X and Y require different proportions of resources.

	A	B	C	Profit ($)
Product X	5	4	35	13
Product Y	15	4	20	23
constraint	480	160	1190	

How to maximize the profit?

- Devote all resources to X: 34 items of X $\Rightarrow$ \$442
- Devote all resources to Y: 32 items of Y $\Rightarrow$ \$736
- 12 items of X, 28 items of Y $\Rightarrow$ \$800

Product Problem

objective function

$$
\begin{array}{lrcrcrl}
 & x & & y & & & \\
\max & 13x & + & 23y & & & \text{Profit} \\
\text{s. t.} & 5x & + & 15y & \le & 480 & A \\
 & 4x & + & 4y & \le & 160 & B \\
 & 35x & + & 20y & \le & 1190 & C \\
 & x & , & y & \ge & 0 &
\end{array}
$$

constraint

decision variable

Standard Form and Slack Form

Standard form of a linear program

"Standard form" of an LP.

- Input: real numbers a_{ij}, c_j, b_i.
- Output: real numbers x_j
- n = # decision variables, m = # constraints.
- Maximize linear objective function subject to linear inequalities.

$$
\begin{array}{lll}
\textbf{(P)} & \max & \sum_{j=1}^{n} c_j x_j \\
 & \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad 1 \le i \le m \\
 & & x_j \ge 0
\end{array}
$$

$$
\begin{array}{lll}
\textbf{(P)} & \max & c^T x \\
 & \text{s.t.} & Ax \le b \\
 & & x \ge 0
\end{array}
$$

Linear. No x^2, $x\,y$, $\arccos(x)$, etc.

Programming. Planning (term predates computer programming).

Slack form

Standard Form.

$$
\begin{array}{llcrcl}
\max & 13x & + & 23y & & \\
\text{s. t.} & 5x & + & 15y & \leq & 480 \\
 & 4x & + & 4y & \leq & 160 \\
 & 35x & + & 20y & \leq & 1190 \\
 & x & , & y & \geq & 0
\end{array}
$$

Slack form.

- Add slack variable for each inequality.
- Now a 5-dimensional problem.

$$
\begin{array}{llcrcccccccl}
\max & 13x & + & 23y & & & & & & & & \\
\text{s. t.} & 5x & + & 15y & + & S_A & & & & & = & 480 \\
 & 4x & + & 4y & & & + & S_B & & & = & 160 \\
 & 35x & + & 20y & & & & & + & S_C & = & 1190 \\
 & x & , & y & , & S_A & , & S_B & , & S_C & \geq & 0
\end{array}
$$

Equivalent forms

Easy to convert variants to slack form.

$$\begin{aligned} \text{(P)}\ \max\ & c^T x \\ \text{s.t.}\ & Ax \le b \\ & x \ge 0 \end{aligned}$$

Less than to equality. $x + 2y - 3z \le 17 \Rightarrow x + 2y - 3z + s = 17,\ s \ge 0$

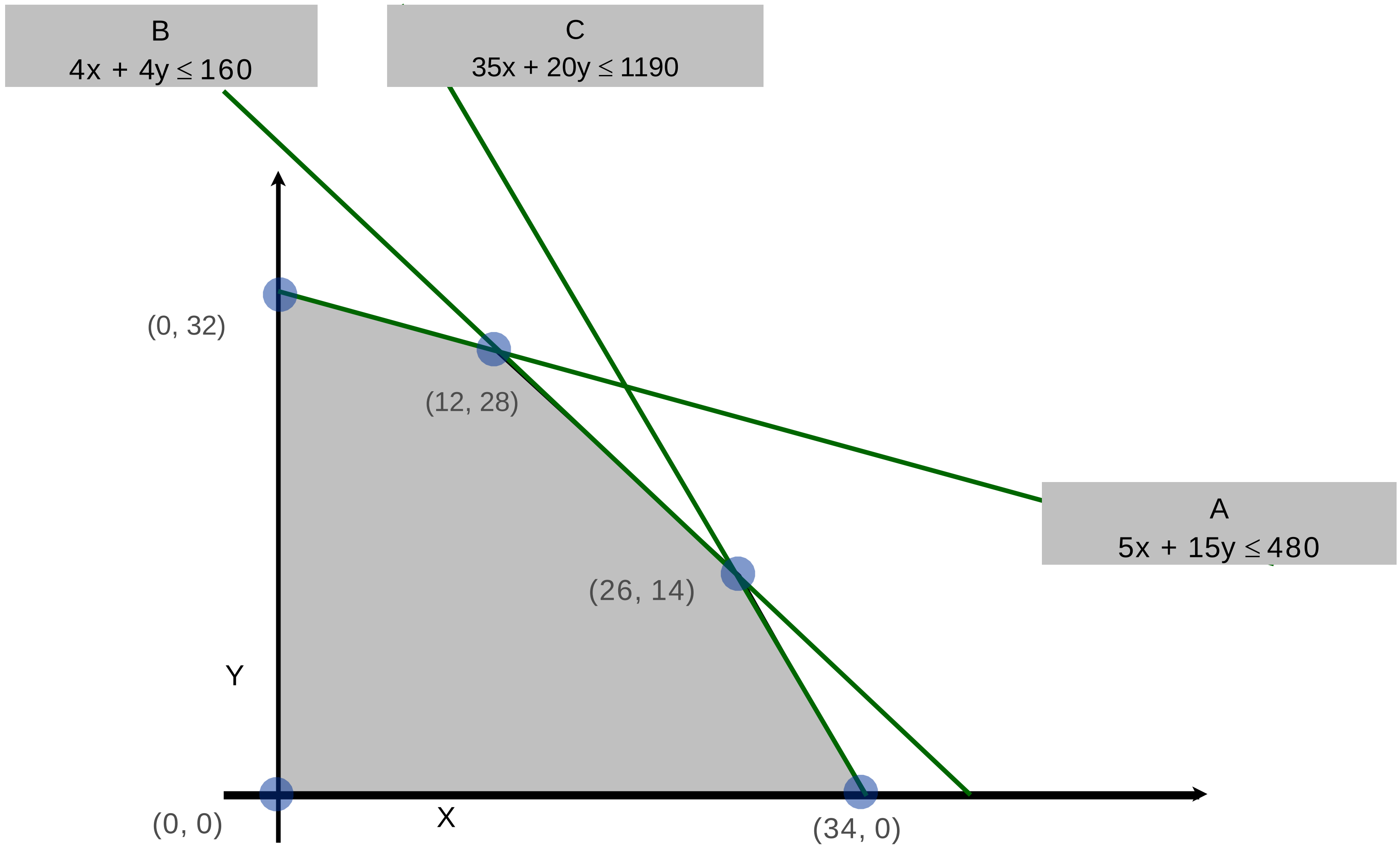
Greater than to equality. $x + 2y - 3z \ge 17 \Rightarrow x + 2y - 3z - s = 17,\ s \ge 0$

Min to max. $\text{Min}\ x + 2y - 3z \Rightarrow \max\ -x - 2y + 3z$

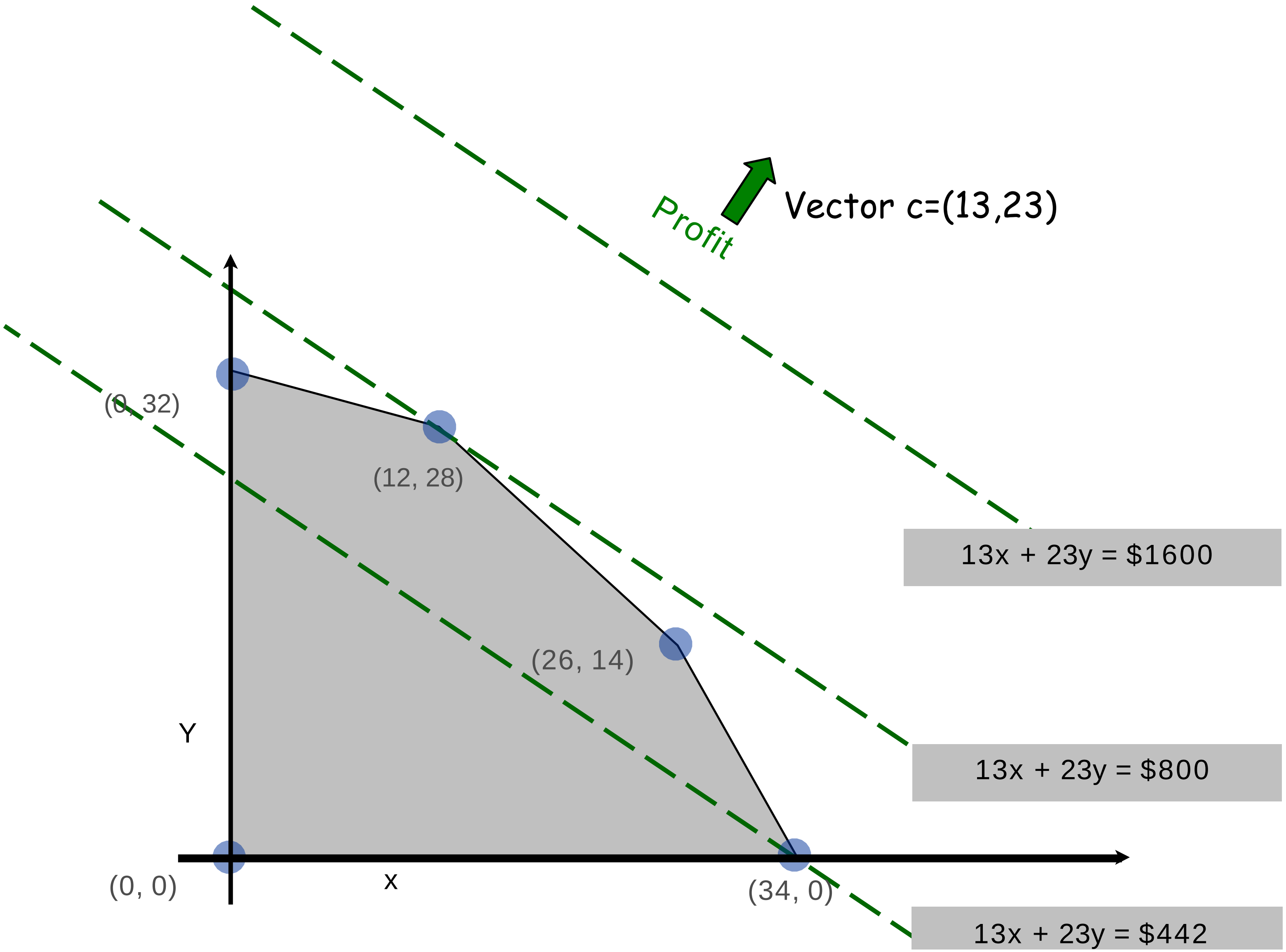
Unrestricted to nonnegative. x unrestricted $\Rightarrow x = x^+ - x^-,\ x^+ \ge 0,\ x^- \ge 0$

Simplex Algorithm

Product problem: feasible region

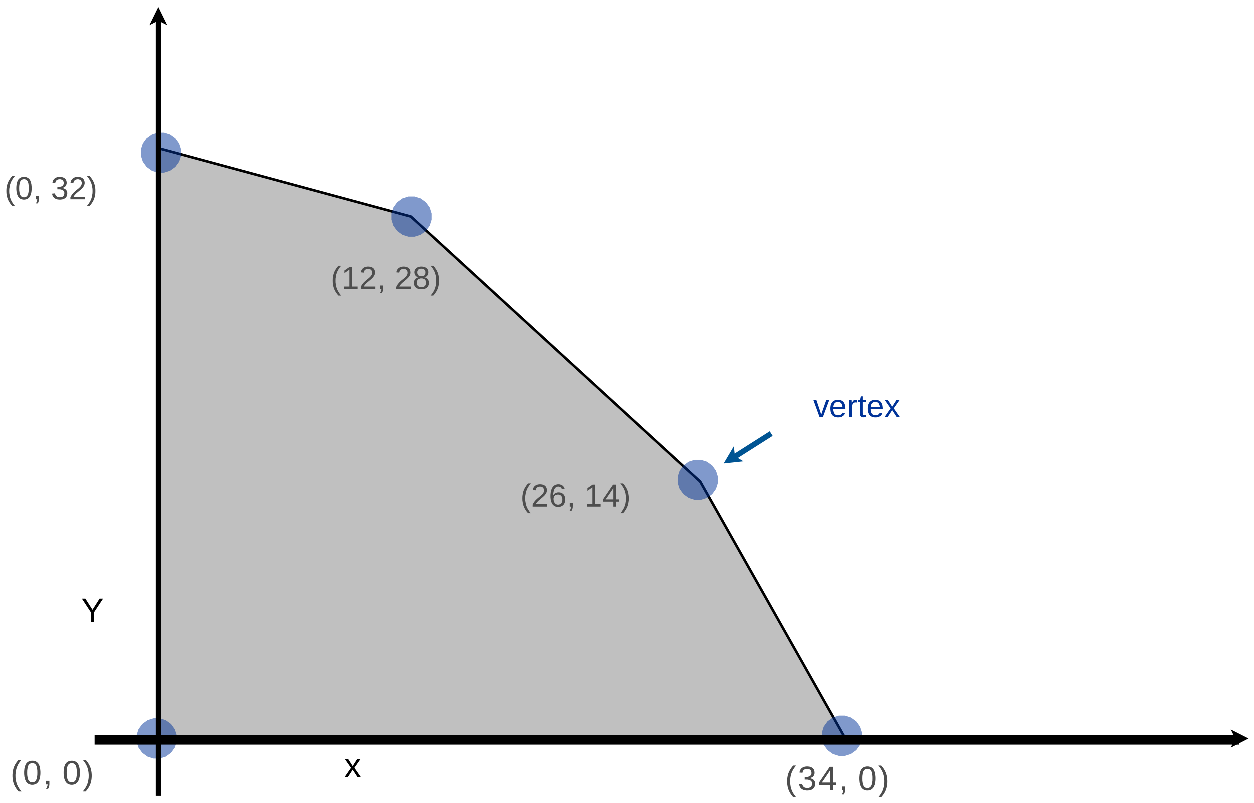


Product problem: objective function



Product problem: geometry

Product problem observation. Regardless of objective function coefficients, an optimal solution occurs at a vertex.



Convexity

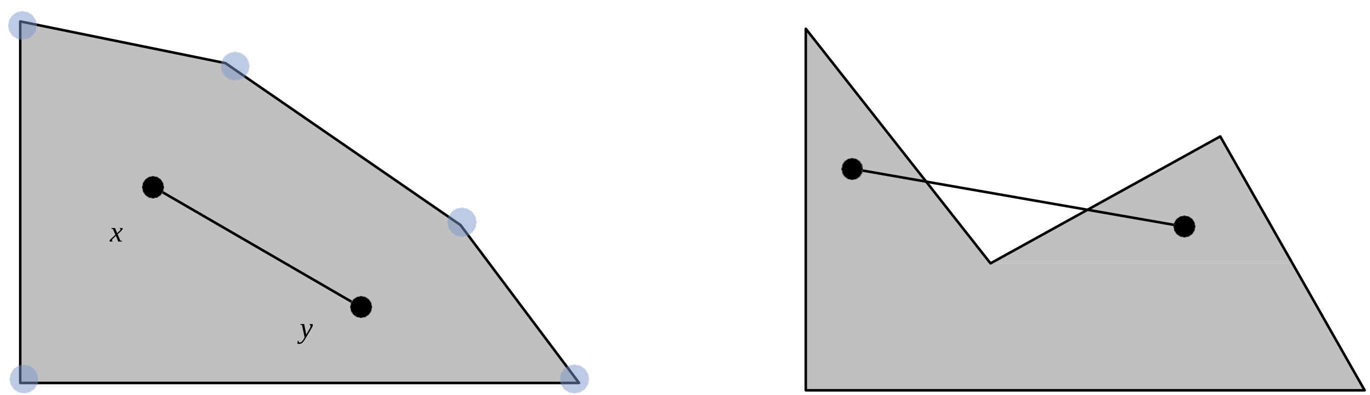
Convex set. If two points x and y are in the set, then so is $\lambda x + (1-\lambda) y$ for $0 \le \lambda \le 1$.

convex combination

not a vertex iff $\exists\, d \neq 0$ s.t. $x \pm d$ in set

Vertex. A point x in the set that can't be written as a strict convex combination of two distinct points in the set.

vertex



convex

not convex

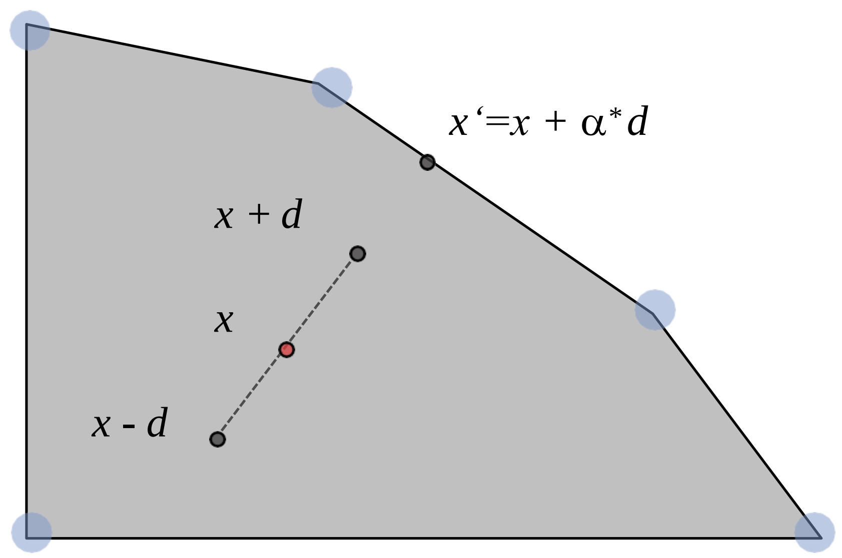
Observation. LP feasible region is a convex set.

Purificaiton

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

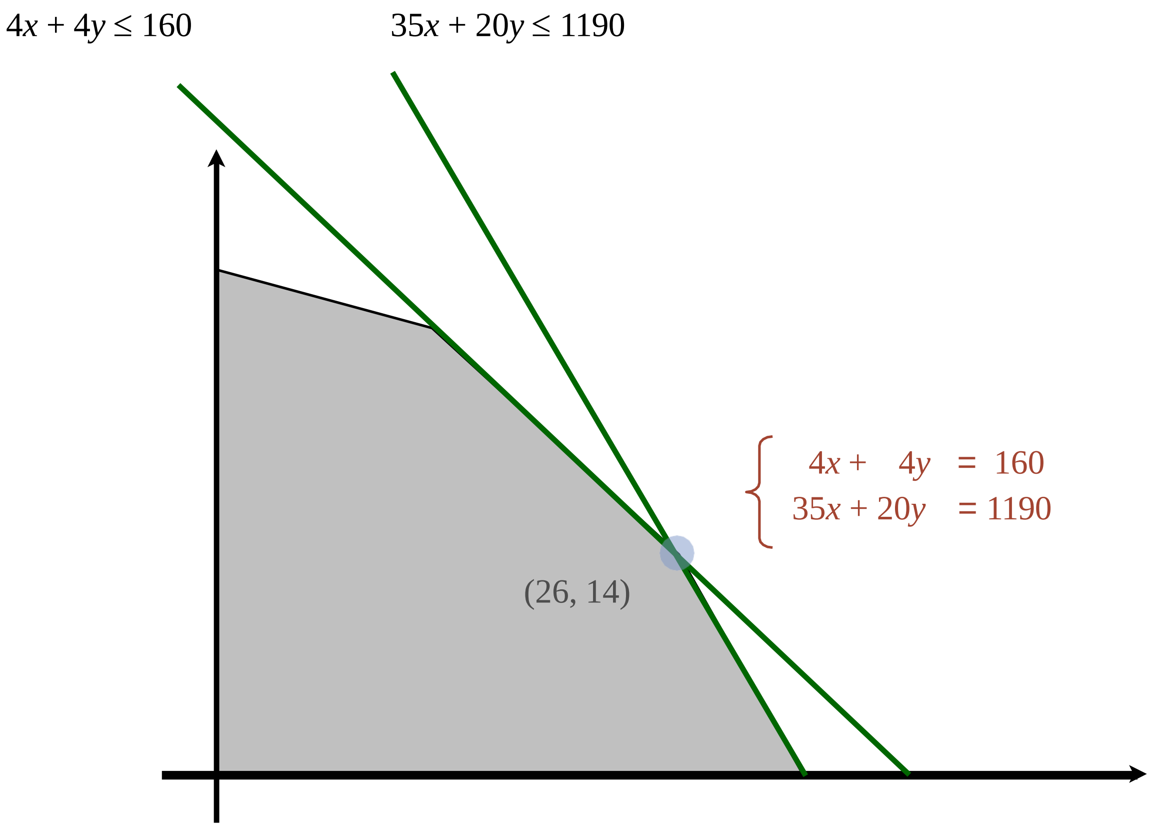
$$\begin{aligned} \textbf{(P)}\ \max\ & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.}\ & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \mathbf{0} \end{aligned}$$

Intuition. If x is not a vertex, move in a non-decreasing direction until you reach a boundary.



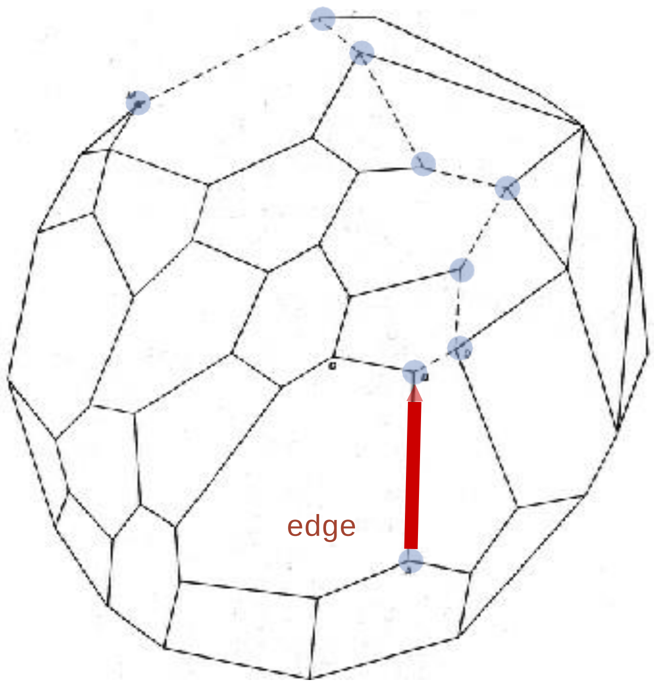
How to find vertices

Theorem. A vertex in $\Re^n$ is uniquely specified by n linearly independent equations.



Simplex algorithm: intuition

Simplex algorithm. [George Dantzig 1947] Move from a vertex to adjacent vertex, without decreasing objective function.



Greedy property. Vertex optimal iff no adjacent vertex is better.

Challenge. Number of vertices can be exponential!

Simplex algorithm works well in practice.

There are other algorithms solving LP in polynomial time.

Duality

LP duality

Primal problem.

$$
\begin{array}{llrcrl}
\text{(P)} & \max & 13x & + & 23y & \\
\text{s. t.} & & 5x & + & 15y & \leq 480 \\
& & 4x & + & 4y & \leq 160 \\
& & 35x & + & 20y & \leq 1190 \\
& & & & x, y & \geq 0
\end{array}
$$

Goal. Find a lower bound on optimal value.

Easy. Any feasible solution provides one.

Ex 1.	(x, y)	$=$	$(34, 0)$	$\Rightarrow$	z^*	≥ 442
Ex 2.	(x, y)	$=$	$(0, 32)$	$\Rightarrow$	z^*	≥ 736
Ex 3.	(x, y)	$=$	$(7.5, 29.5)$	$\Rightarrow$	z^*	≥ 776
Ex 4.	(x, y)	$=$	$(12, 28)$	$\Rightarrow$	z^*	≥ 800

LP duality

Primal problem.

$$
\begin{array}{llcl}
\text{(P)} & \max & 13x + 23y & \\
\text{s. t.} & & 5x + 15y & \le 480 \\
& & 4x + 4y & \le 160 \\
& & 35x + 20y & \le 1190 \\
& & x, y & \ge 0
\end{array}
$$

Goal. Find an upper bound on optimal value.

Ex 1. Multiply 2nd inequality by 6: $24x + 24y \le 960$.

$$\Rightarrow \quad z^* = \underbrace{13x + 23y}_{\text{objective function}} \le 24x + 24y \le 960.$$

LP duality

Primal problem.

$$
\begin{array}{llcl}
\text{(P)} & \max & 13x + 23y & \\
\text{s. t.} & & 5x + 15y & \leq 480 \\
 & & 4x + 4y & \leq 160 \\
 & & 35x + 20y & \leq 1190 \\
 & & x, y & \geq 0
\end{array}
$$

Goal. Find an upper bound on optimal value.

Ex 2. Add 2 times 1st inequality to 2nd inequality:

$$\Rightarrow \quad z^* = 13x + 23y \;\leq\; 14x + 34y \;\leq\; 1120.$$

LP duality

Primal problem.

$$
\begin{array}{llcccl}
(\text{P}) & \max & 13x & + & 23y & \\
\text{s. t.} & & 5x & + & 15y & \le 480 \\
 & & 4x & + & 4y & \le 160 \\
 & & 35x & + & 20y & \le 1190 \\
 & & x, y \ge 0 & & &
\end{array}
$$

Goal. Find an upper bound on optimal value.

Ex 2. Add 1 times 1st inequality to 2 times 2nd inequality:

$$\Rightarrow \qquad z^* = 13x + 23y \le 13x + 23y \le 800.$$

Recall lower bound. $(x, y) = (12, 28) \Rightarrow z^* \ge 800$

Combine upper and lower bounds: $z^* = 800$.

LP duality

Primal problem.

$$\begin{array}{llllll}
\text{(P)} & \max & 13x & + & 23y & \\
\text{s. t.} & & 5x & + & 15y & \leq 480 \\
 & & 4x & + & 4y & \leq 160 \\
 & & 35x & + & 20y & \leq 1190 \\
 & & x, y \geq 0 & & &
\end{array}$$

Idea. Add nonnegative combination (A, B, C) of the constraints s.t.

$$\begin{aligned}
13x + 23y &\leq (5A + 4B + 35C)\,x + (15A + 4B + 20C)\,y \\
&\leq 480A + 160B + 1190C
\end{aligned}$$

Dual problem. Find best such upper bound.

$$\begin{array}{lllllllll}
\text{(D)} & \min & 480A & + & 160B & + & 1190C & & \\
 & \text{s. t.} & 5A & + & 4B & + & 35C & \geq & 13 \\
 & & 15A & + & 4B & + & 20C & \geq & 23 \\
 & & A & , & B & , & C & \geq & 0
\end{array}$$

LP duals

Canonical form.

$$
\text{(P)} \quad \max\ c^T x \quad \text{s. t.} \quad Ax \le b, \quad x \ge 0
$$

$$
\text{(D)} \quad \min\ y^T b \quad \text{s. t.} \quad A^T y \ge c, \quad y \ge 0
$$

Double dual

Canonical form.

$$
\begin{aligned}
\text{(P)}\quad & \max\ c^T x \\
& \text{s. t. } Ax \le b \\
& \quad x \ge 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\text{(D)}\quad & \min\ y^T b \\
& \text{s. t. } A^T y \ge c \\
& \quad y \ge 0
\end{aligned}
$$

Property. The dual of the dual is the primal.

Pf. Rewrite (D) as a maximization problem in canonical form; take dual.

$$
\begin{aligned}
\text{(D')}\quad & \max\ -y^T b \\
& \text{s. t. } -A^T y \le c \\
& \quad y \ge 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\text{(DD)}\quad & \min\ -c^T z \\
& \text{s. t. } -(A^T y)^T \ge -b \\
& \quad z \ge 0
\end{aligned}
$$

Taking duals

LP dual recipe.

Primal (P)	maximize	minimize	Dual (D)
constraints	$a x = b_i$	unrestricted	variables
	$a x \leq b$	$y_i \geq 0$	
	$a x \geq b_i$	$y_i \leq 0$	
Variables	$x_j \geq 0$	$a^T y \geq c_j$	constraints
	$x_j \leq 0$	$a^T y \leq c_j$	
	unrestricted	$a^T y = c_j$	

Pf. Rewrite LP in standard form and take dual.

LP strong duality

Theorem. [Gale-Kuhn-Tucker 1951, Dantzig-von Neumann 1947]
For $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, if (P) and (D) are nonempty, then max = min.

$$
\begin{array}{lll}
\text{(P)} & \max c^T x & \\
& \text{s. t.} & Ax \le b \\
& & x \ge 0
\end{array}
\qquad
\begin{array}{lll}
\text{(D)} & \min y^T b & \\
& \text{s. t.} & A^T y \ge c \\
& & y \ge 0
\end{array}
$$

LP weak duality

Theorem. For $A \in \Re^{m \times n}, b \in \Re^m, c \in \Re^n$, if (P) and (D) are nonempty, then max $\leq$ min.

$$\text{(P)} \quad \max c^T x \quad \text{s. t.} \quad Ax \leq b, \quad x \geq 0$$

$$\text{(D)} \quad \min y^T b \quad \text{s. t.} \quad A^T y \geq c, \quad y \geq 0$$

Pf. Suppose $x \in \Re^n$ is feasible for (P) and $y \in \Re^m$ is feasible for (D)

$$y \geq 0, \quad Ax \leq b \quad \Rightarrow \quad y^T A x \leq y^T b$$

$$x \geq 0, \quad A^T y \geq c \quad \Rightarrow \quad y^T A x \geq c^T x$$

Combine: $c^T x \leq y^T A x \leq y^T b$. •

LP duality: Example

Product problem: find optimal mix of product X and product Y to maximize profit.

$$
\begin{aligned}
\text{(P)} \quad \max \quad & 13x + 23y \\
\text{s. t.} \quad & 5x + 15y \leq 480 \\
& 4x + 4y \leq 160 \\
& 35x + 20y \leq 1190 \\
& x, y \geq 0
\end{aligned}
$$

$x^* = 12$
$y^* = 28$
$OPT = 800$

Dual: buy individual resources at min cost.

$$
\begin{array}{llcccccc}
\text{(D)} & \min & 480A & + & 160B & + & 1190C & \\
& \text{s. t.} & 5\text{A} & + & 4B & + & 35C & \geq 13 \\
& & 15\text{A} & + & 4B & + & 20C & \geq 23 \\
& & & , & B & , & C & \geq 0
\end{array}
$$

$A^* = 1$
$B^* = 2$
$C^* = 0$
$OPT = 800$

Modeling Problems by LP

Shortest Path

- Give a weighted, directed graph $G = (V,E)$ and a source s and destination t.
- Compute d[t] which is the weight of a shortest path from s to t.

To express this problem as a linear program:

- we need to determine a set of variables and constraints that define when we have a shortest path from s to t.

We exploit the Bellman-Ford algorithms

```
min  d[t]
s.t. d[v] ≤ d[u]+w(u,v) for each edge (u,v) ∈ E
     d[s] = 0
     d(v] ≥ 0 for all v ∈ V
```

- #variables = #vertices
- #constrains = #edges

Dual

Primal Problem.

```
max   -d[t]
s.t.  d[v] - d(u) ≤ w(u,v) for each edge (u,v) ∈ E
      d[s] = 0
      d(v] ≥ 0 for all v ∈ V
```

Dual Problem again is the shortest path problem; just modeling in a different way.

min $\sum x_{(u,v)} w(u,v)$

s.t. $\sum x_{(u,v)} - \sum x_{(v,u)} \geq 0$ for each vertex v$\neq s,t,$ $\in$ E

$\sum x_{(t,u)} - \sum x_{(u,t)} \geq 1$

$\sum x_{(u,s)} - \sum x_{(s,u)} \geq 1$

$x_{(u,t)} \geq 0$ for all (u,v) $\in$ E

Max Flow

- Give a weighted, directed graph G = (V,E) and a source s and sink t.
- Compute max flow from s to t.

```
Max    ∑ f(s,u) − ∑ f(u,s)
s.t. f(u,v) ≤ c(u,v) for each edge (u,v) ∈ E
       ∑_u f(u,v) − ∑_u f(v,u) = 0 for all v ∈ V-{s,t}
```

- #variables = #edges
- #constrains = #edges + #vertices

Dual

Primal Problem.

```
Max    ∑f(s,u) − ∑f(u,s)
s.t.  f(u,v) ≤ c(u,v) for each edge (u,v) ∈ E
       ∑_u f(u,v) − ∑_u f(v,u) = 0 for all v ∈ V-{s,t}
```

Dual Problem. Do it as exercise.

```
min    ?
s.t.  ?
```

Another LP for Max Flow

Primal Problem.

$$\begin{array}{ll} \texttt{max} & \sum x_p \\ \texttt{s.t.} & \sum_{p:(u,v)\in p} x_p \le c(u,v) \;\; \texttt{for all (u,v)} \in \texttt{E} \\ & x_p \ge 0 \end{array}$$

Dual Problem. Min cut.

$$\begin{array}{ll} \texttt{min} & \sum_{(u,v)\in E} y_{(u,v)} c(u,v) \\ \texttt{s.t.} & \sum_{(u,v)\in p} y_{(u,v)} \ge 1 \;\; \texttt{for all} \; p \in P \\ & y_{(u,v)} \ge 0 \end{array}$$

Strong duality theorem says max flow = min cut (a new proof)

References

References

- Chapter 29 of the text book "introduction to algorithms" by CLRS, 3rd edition.
- The original slides were prepared by Kevin Wayne.